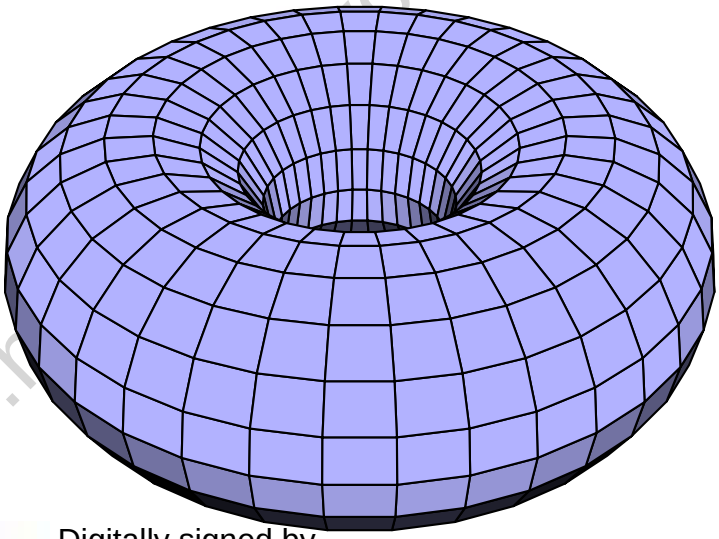
Advanced Topics in Mathematical Physics

By
Peter Jones



A
Dysprosium Project

Advanced Topics in Mathematical Physics

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Dedicated to Cynthia and Mercedes

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The Author thanks Carlo DiJuliani.

Contents

Preface

I like to see the mathematics of physics to be learnt in this way. I liked it this way when I was a freshman. For example, I encountered frequently the term, "algebra" in different contexts with much confusion what did they mean and if they were the same thing. I was confused about a measure, a measurable set and a measure space. I could not appreciate how a mapping is related to a parametric description of a surface and the concept of a manifold. I could not see clearly, why partitioning of a set with a relation is called an equivalence class and why a mapping is called functional and not just a function. I did not know that $(\mathbb{R}/\mathbb{Z})$ and $(\mathbb{R}/\mathbb{Z})$ are different, but the same notation is used for them. I liked, average students with less commitment to mathematics could fall on the track to appreciate more beautiful sides of the story sooner rather than later. It is not encyclopaedically stupid. Yet, it tries not to leave a loop hole behind from those mathematics object that one needs in studying physics, to look at them like a mathematician. Besides these, a good study of physics implies that one recognises, in current technology, what is the physical size of a magnet that produces a one Milli-Tesla or a ten Milli-Tesla or a point one Tesla field; such things. Peter Jones

Part 1

Foundations

CHAPTER 1

Set Theory

I assume that the reader has a good understanding of set theory material studied in his first course in freshman calculus. Here we develop a brief course on some advanced topics in set theory as it is taught to students of mathematics. We begin it with familiarity with union and intersection on class of sets. Then we discuss degenerate cases in sets which need to work with empty set.

1.1. Operations over a set

DEFINITEION 1.1.1. *Union over a Set : Union over a set is the generalization of union of sets, that fuses all members of a set together. These members are usually sets which are constituted of other elements.*

$$\bigcup \mathcal{A} = \{x \mid \exists a \in \mathcal{A} \ni x \in a\}$$

$$\bigcup \varnothing = \varnothing$$

DEFINITEION 1.1.2. *Intersection Over a Set : Intersection over a set is generalization of intersection of sets that leaves out those elements common to all members of that set. These members are usually sets which are constituted of other elements.*

$$\bigcap \mathcal{A} = \{x \mid \forall a \in \mathcal{A} \Rightarrow x \in a\}$$

In contrast to union, intersection over an empty set has no conclusive meaning and we have to define the intersection for the empty set as,

$$\bigcap \varnothing \triangleq \varnothing$$

1.2. Power Sets

I assume reader is already familiar with the idea of power set. To refresh that idea I define,

DEFINITEION 1.2.1. *Power Set : It is defined as the set of all subsets of any set $\mathcal{A}$. Precisely,*

$$\mathscr{P}\mathcal{A} = \{x \mid x \subseteq \mathcal{A}\}.$$

And also we have,

$$\mathscr{P}\varnothing = \{\varnothing\}$$

Hence, if $x \subseteq \mathcal{A}$ we, then, can write $x \in \mathscr{P}$. Please pay attention to belongness $\in$ symbol here. Also as an exercise note that $\bigcup \mathscr{P}\mathcal{A} = \mathcal{A}$.

Remark 1.2.1. We know what is a power set. Frequently we need to select certain collections of subsets of a set with certain structure out of the entire collection of subsets. For example $\mathfrak{M}$ which is a subcollection of $\mathscr{P}$. That is, $\mathfrak{M} \subseteq \mathscr{P}$. When we freely select an arbitrary collection and like to impose certain structure to them we call that collection a free collection and we show it by $\mathscr{F}$. To impose the certain structure to this collection $\mathscr{F}$, we make an intersection over all those collections that have that structure and contain $\mathscr{F}$ as a subset. Then we have the **smallest** collection shown say by $\mathscr{F}^*$ that is endowed with our desired structure. We easily can verify that having any two sets in $\mathscr{F}^*$ then we have their intersection in $\mathscr{F}^*$. Also if a set belongs to $\mathscr{F}^*$ then all of its subsets also belong to $\mathscr{F}^*$.

EXERCISE 1. *Convince yourself that these three sets $\varnothing$, and $\{\varnothing\}$, and $\{\{\varnothing\}\}$ are different sets; none of them equal to others.*

EXERCISE 2. *Use one of the $\subseteq$ or $\in$ or both in place of dots in the following:*

(1) $\{\varnothing\} \cdots \{\varnothing, \{\varnothing\}\}$
(2) $\{\varnothing\} \cdots \{\varnothing, \{\{\varnothing\}\}\}$
(3) $\{\{\varnothing\}\} \cdots \{\varnothing, \{\varnothing\}\}$
(4) $\{\{\varnothing\}\} \cdots \{\varnothing, \{\{\varnothing\}\}\}$
(5) $\{\{\varnothing\}\} \cdots \{\varnothing, \{\varnothing, \{\varnothing\}\}\}$

EXERCISE 3. *Simplify*

(1) $\bigcap\{\mathscr{P}\mathscr{P}\mathscr{P}\varnothing, \mathscr{P}\mathscr{P}\varnothing, \mathscr{P}\varnothing, \varnothing\}$.
(2) $\bigcap\{\mathscr{P}\mathscr{P}\mathscr{P}\{\varnothing\}, \mathscr{P}\mathscr{P}\{\varnothing\}, \mathscr{P}\{\varnothing\}\}$.

EXERCISE 4. *Let A be the set $\{\{\varnothing\}, \{\{\varnothing\}\}\}$ Evaluate the following:*

$$(a)\ \mathscr{P}A \quad (b)\ \bigcup A \quad (c)\ \mathscr{P}\bigcup A \quad (d)\ \bigcup \mathscr{P}A$$

EXERCISE 5. *Show that for every set A we have, $\{\varnothing, \{\varnothing\}\} \in \mathscr{P}\mathscr{P}\mathscr{P}A$.*

1.3. Empty Set Degenerate Cases

It is interesting to chase the last remark of the previous section in building further sets by getting the power sets of empty set, such as

$$\mathscr{P}\mathscr{P}\varnothing \quad \text{or} \quad \mathscr{P}\mathscr{P}\mathscr{P}\varnothing \quad \text{or} \quad \mathscr{P}\mathscr{P}\mathscr{P}\mathscr{P}\varnothing \ ...$$

A clever way of making more sets is to assume that we have a beginning set S_0 and make a pyramid of sets using only S_0 and power sets made thereby such that each lower step includes the higher step, in this way

$$S_0$$

$$S_1 = S_0 \cup \mathscr{P}S_0$$

$$S_2 = S_1 \cup \mathscr{P}S_1$$

$$\vdots$$

$$S_{n+1} = S_n \cup \mathscr{P}S_n$$

$$\vdots$$

You can notice that,

$$S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_{n+1} \subset \cdots$$

Please note that the number of elements in each set S_{n+1} is finitely limited. We could continue building up any set with diabolically any number of elements but still we have a finite set. To overcome that we make the following set,

$$S_\omega = S_0 \cup S_1 \cup S_2 \cup \cdots \cup S_{n+1} \cup \cdots$$

That set has countably infinite number of elements

Definiteion 1.3.1. *Successor: For any set x, its successor x^+ is defined as the set*

$$x^+ = x \cup \{x\}$$

Definiteion 1.3.2. *Inductive Set: A set A is said to be inductive if it meets both of these conditions:*

(1) $\varnothing \in A$.
(2) *if* $x \in A$ *then* $x^+ \in A$.

1.3.1. Natural Number System. The exciting application of above observation is building natural numbers system from the beginning empty set $\varnothing$. We define,

$$\begin{aligned} 0 &= \{\} \\ &= \varnothing \end{aligned}$$

0-set has no element. Then

$$\begin{aligned} 1 &= 0^+ \\ &= 0 \cup \mathscr{P}0 \\ &= \varnothing \cup \{\varnothing\} \\ &= \{\varnothing\} \\ &= \{0\} \end{aligned}$$

1-set has one element. Then

$$\begin{aligned} 2 &= 1^+ \\ &= 1 \cup \mathscr{P}1 \\ &= \{\varnothing\} \cup \{\varnothing, \{\varnothing\}\} \\ &= \{\varnothing, \{\varnothing\}\} \\ &= \{0, 1\} \end{aligned}$$

2-set has two elements. And

$$\begin{aligned} 3 &= 2^+ \\ &= 2 \cup \mathscr{P}2 \\ &= \{\varnothing, \{\varnothing\}\} \cup \{\varnothing, \{\varnothing, \{\varnothing\}\}, \varnothing, \{\varnothing\}\} \\ &= \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} \\ &= \{0, 1, 2\} \end{aligned}$$

3-set has three elements. And finally,

$$\begin{aligned} &\vdots \\ n+1 &= n^+ \\ &= n \cup \mathscr{P}n \\ &= \{0, 1, 2, 3, \cdots, n\} \end{aligned}$$

n-set has n elements. Here again we can make the infinite set of natural numbers shown by ω.

$$\omega = 0 \cup 1 \cup 2 \cup 3 \cup \cdots \cup n \cup \cdots$$

which is an inductive set.

Definiteion 1.3.3. *Arithmetic Addition : Addition of two natural numbers m and n is shown by symbol + and defined as,*

$$m + n \triangleq m \cup n.$$

Arithmetic multiplication is just an addition repeated many times.

$$m.n \triangleq \underbrace{m \cup m \cup \cdots \cup m}_{for\ n\ times.}$$

1.4. Mappings

An ordered pair is a set theory concept defined carefully to be constructed out of two elements of sets such that if the position of elements in their arrangements changes their assigned meaning changes. An ordered pair of two elements $x \in X$ and $y \in Y$ is shown with notation (x, y). More concisely,

Definiteion 1.4.1. *Ordered Pair : An ordered pair is a set such that,*

(1) *Generally $(x, y) \neq (y, x)$ for $x \in X$ and $y \in Y$.*
(2) *If $(x, y) = (v, w)$ for some x , $v \in X$ and some y , $w \in Y$, then $x = v$ and $y = w$.*

It is interesting to know that in abstract mathematics an ordered pair (x, y) is defined as equal to the set $\{\{x\}, \{x, y\}\}$ such that it satisfies uniquely the requirements of the above definition (proposed by K. Kuratowski, 1921).

Remark 1.4.1. Please look at these interesting corollaries. You can convince yourself.

$$\bigcup (x, y) = \{x, y\}$$

$$\bigcup (X \times Y) = X \cup Y$$

$$\bigcap (x, y) = \{x\}$$

$$\bigcap (X \times Y) = X$$

Definiteion 1.4.2. *Cartesian Products : Cartesian product of two sets X and Y is the set of all **ordered** pairs $\{\forall\ (x, y) \,|x \in X \text{ and } y \in Y\}$. In particular we can define the Cartesian product of $X \times X$.*

Remark 1.4.2. Cartesian product of more than two sets such as X, Y, and Z can be defined in a similar fashion with an important warning: this product is not generally associative, except that we have Cartesian product of the same set as follows.

We can extend the Cartesian product to $X \times X \times \cdots \times X \times X$ such that we can define X^n for a set X.
There are degenerate cases worth of some attention.

(1) The product $X \times \varnothing$. This is equal to the set of singletons $(x,\ x)$. To see that we write, $(x,\ x) = \{\{x\}, \{x, x\}\} = \{\{x\}, \{x\}\} = \{\{x\}\}$. Now, one element of product $X \times \varnothing$ can be written as $\{\{x\}, \{x,\ \}\} = \{\{x\}, \{x\}\} = \{\{x\}\}$. We left a blank space after x, to show that we have selected an element from empty set as the second component of the ordered pair. We can call this as an **ordered single** and show it by (x). Note that this is $\{\{x\}\}$ and is different from $\{x\}$.
(2) The product $\varnothing \times \varnothing$. This is considered special case of the above and is equal to the set $\{\{\ \}\} = \{\varnothing\}$. We know that this is not an empty set anymore and is different from just an empty set $\{\ \} = \varnothing$. We can call it an **ordered empty** (in some contexts [**13**] it is called an empty list) and show it by $(\ )$ or in some contexts by Λ.
(3) The product $\varnothing \times X$. This, if you prefer, can only be defined as the empty set $\varnothing$. We cannot find anything out of the set $\{\varnothing,\ \{x\}\}$. It has not any meaning.
(4) In X^n is it possible to have $n = 0$?
Answer is affirmative. We define it as the set of all ordered empties, $X^0 \triangleq \{(\ )\}$[**7**]. Hence, $X^0 = \{\varnothing \times \varnothing\}$. We show $X^0 = \{\Lambda\}$, and we call it empty product. This Λ set notation later will get other usages.
(5) In X^n is it possible to have $n = 1$?
Answer is affirmative. We define $X^1 = \{(x)\}$, the set of all ordered singles. Hence, $X^1 = X \times \varnothing$. Please differentiate X^1 with X. That is $X^1 \neq X$. If, for example, $X = \{a, b, c\}$ then $X^1 = \{(a), (b), (c)\}$.
(6) In X^n is it possible to have $n = \infty$ countably?
Answer is again affirmative. We study such products later.
(7) In X^n is it possible to have $n = \infty$, but uncountable?
Answer is again affirmative. We later define the Cartesian product for any value of n.

Definiteion 1.4.3. *Relation : A relation R from a set X to another set Y is just any subset of $X \times Y$. When $x \in X$ is related through R to $y \in Y$ we show it as xRy, or $(x,\ y) \in R$.*

Note that the subset mentioned in above definition is selected on an arbitrary appropriateness. That is, any arbitrary selection of any subset of $X \times Y$ is said to be a relation from X to Y.

Example 1.4.1. *Incidence (in Projective Geometry): Assume $\mathcal{P}$ is the set of points and $\mathcal{L}$ is the set of lines in a Euclidean plane (more beautifully in a division*

ring R), a subset of $\mathcal{P} \times \mathcal{L}$ is called an ***incidence*** *if that relation satisfies set of axioms of projective geometry, as they follow.*

(1) $\mathcal{L} \neq \varnothing$ *(we have, at least, one line).*
(2) $\exists a, b, c \in \mathcal{P}, \ni a, b, c \in l, \forall l \in \mathcal{L}$ *(on each line we have at least three points).*
(3) $\forall l \in \mathcal{L} \exists p \in \mathcal{P} \ni p \notin l$ *(taking any line, we have at least one point not on that line).*
(4) $\forall l \in \mathcal{L} \exists p \in \mathcal{P} \ni p \notin \mathcal{L}$ *(there is one point not belonging to any line; such a point is called an ideal point).*
(5) $\forall l_1$ *and* $l_2 \in \mathcal{L} \exists p \in \mathcal{P} \ni p \in l_1$ *and* $p \in l_2$ *(any two lines intersect at a point, even if you believe those lines are parallel; they might intersect at the ideal point in this case).*

DEFINITEION 1.4.4. *Graph of a Relation : Assume $R \subset X \times Y$ is a relation. Then the set $\{(x, y) \,|\, x \in X$ and $y \in Y\}$ is called the graph of R.*

Mapping is a relation R from a set X to another set Y, where for each element of X we assign only one unique element in Y. Instead of letter R, we prefer to usually use the letter f for a mapping. More concisely,

DEFINITEION 1.4.5. *Mapping : A mapping f of set X to set Y is a relation from X to Y such that $\forall x \in X$ there is only a unique $y \in Y$ that satisfies xfy or $(x, y) \in f$. We use notations $f : X \longrightarrow Y$, reads as mapping f from X to Y, and $x \mapsto y$ reads x maps to y for showing a mapping. Finally, we write $y = f(x)$ and read it as y is the map of x under f. Usually we write them in a stacked form as,*

$$\begin{aligned} f : X &\longrightarrow Y \\ x &\mapsto y \\ y &= f(x) \end{aligned}$$

The last notation is also read as y is the ***value*** *of f at x. In, yet another wording we say f takes x to y. We say y is* ***the*** *image of x under mapping f and x is* ***a*** *pre-image of y with respect to f.*

A mapping frequently is termed as a **function**. We keep the usage of term "function" for mappings to the set of **real** numbers $\mathbb{R}$, or to the set of **complex** numbers $\mathbb{C}$, or their subsets all through this book. Hence we call a **real valued** or a **complex valued** mapping a real function or a complex function, respectively. Later we see that a mapping from a vector space to its *scalar* field is usually called a **functional**. This term is kept as it is, due to historical usage, though it could be avoided.

Remark 1.4.3. Unique : To clear the meaning of **unique** in definition of mapping we can, **alternatively** say that if we take two different elements y_1 such that $y_1 = f(x_1)$ and y_2 such that $y_2 = f(x_2)$ and we find out that $y_1 \neq y_2$ then to have a mapping it is necessary that $x_1 \neq x_2$. If any point x in X maps to more than one point y in Y then f is **not** a mapping anymore. It will be a relation. This is how we check if a relation is a mapping. Note that it is possible in a mapping that different x's in X map to the same y in Y, inverse is not true.

To summarize, assume
$y_1 = f(x_1)$ and $y_2 = f(x_2)$ then,

$$if\ y_1 \neq y_2 \qquad \text{then} \quad x_1 \neq x_2,$$

alternatively,

$$if\ x_1 = x_2 \qquad \text{then} \quad y_1 = y_2;$$

on the other hand,

$$if\ y_1 = y_2 \quad \text{then it could be that} \quad x_1 \neq x_2,$$

alternatively,

$$if\ x_1 \neq x_2 \quad \text{then it could be that} \quad y_1 = y_2;$$

See Figure 1.1

DEFINITEION 1.4.6. *Domain of a Mapping : Assume $f : X \longrightarrow Y$ is a mapping, then the set X is defined as the* ***domain*** *of the mapping f.*

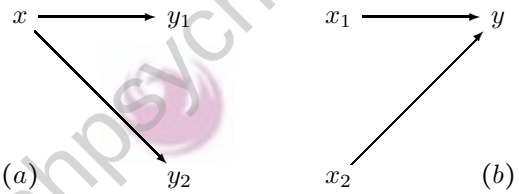


FIGURE 1.1. (a) cannot be a mapping. (b) is a mapping.

DEFINITEION 1.4.7. *Co-domain of a Mapping : Assume $f : X \longrightarrow Y$ is a mapping, then the set Y is defined as the* ***co-domain*** *of the mapping f.*

In a mapping f the set of those points $y \in Y$ such that y is the image of some points $x \in X$ constitute a subset $A \subseteq Y$. This subset is the range of the mapping. More concisely, we have,

DEFINITEION 1.4.8. *Range of a Mapping : Assume $f : X \longrightarrow Y$ is a mapping, then the set $\forall y \in Y$ such that $\exists x \in X$ and $y = f(x)$ is a subset $A \subseteq Y$. This subset is the* ***range*** *of the mapping and is shown by notation $f[X]$. See Figure 1.2*

Range of a mapping f is also called the **set of values** of the mapping f.
In a mapping we have three objects to recognise, domain, range and f. The latter, f is also a set. We have, $f \subset X \times Y$. Or more concisely,

$$f = \{(x, y) \,|\, y = f(x) \text{ where } x \in X \text{ and } y \in Y\}.$$

DEFINITEION 1.4.9. *Image of a Mapping : Assume $f : X \longrightarrow Y$ is a mapping, and $E \subseteq X$ then the set $\forall y \in Y$ such that $x \in E$ and $y = f(x)$ is a subset $S \subseteq Y$. This subset is the* ***image*** *of the set E and is shown by notation $f[E]$.*

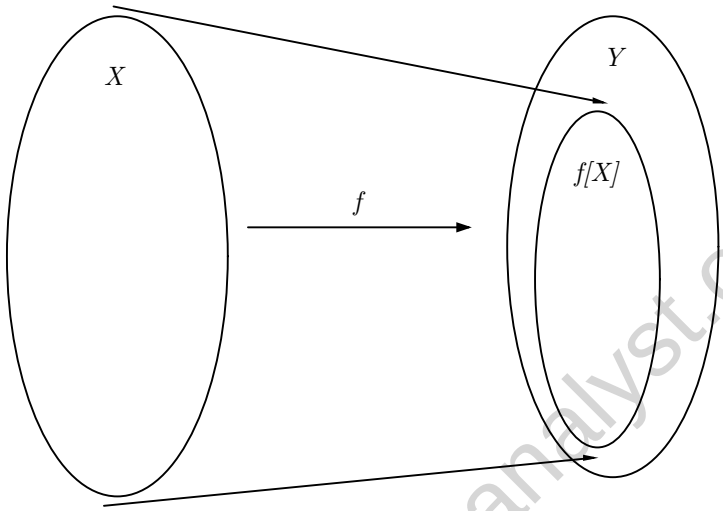


FIGURE 1.2. Range of Set X under Mapping f into Set Y.

It is necessary to recognise the image of a mapping from the *graph of a mapping.*

DEFINITEION 1.4.10. *Graph of a mapping : This set frequently is known as the graph of the mapping and is described as,*

$$graph(f) = \{(x,\, f(x)) \,|\, x \in X \text{ and } f(x) \in Y\}\,.$$

DEFINITEION 1.4.11. *Inverse Image of a Mapping : Assume $f : X \longrightarrow Y$ is a mapping, and $S \subseteq Y$ then the set $\forall x \in X$ such that $y \in S$ and $y = f(x)$ is a subset $E \subseteq X$. This subset E is the* ***inverse image*** *of the set S and is shown by notation $f^{-1}[S]$. We frequently, call $f^{-1}[S]$ as the* ***pre-image*** *of S.*

If $y \in Y$ then you should recognize between $x = f^{-1}(y)$ where $x \in X$ and $E = f^{-1}[\{y\}]$ where $E \subseteq X$. Though it is true that frequently $f^{-1}(y)$ gives more than one $x \in X$ and some people might have that in mind when they interchangeably use the same notation for both $f^{-1}(y)$ and $f^{-1}[\{y\}]$.
You might notice that f^{-1} is not necessarily a mapping for itself, and it could be only a relation not a mapping.

Example 1.4.2. *Assume the set $X = \{\varnothing, \{\varnothing\}\}$ and the set $Y = \{a, b\}$. We define $f : X \to Y$ as $f = \{(\varnothing, a), (\{\varnothing\}, b)\}$. Please check that $f[\{\varnothing\}] = \{a\}$, and $f(\{\varnothing\}) = b$.*

DEFINITEION 1.4.12. *Surjective Mappings : Assume $f : X \longrightarrow Y$ is a mapping. If for each $y \in Y$ we can find* ***at least*** *one (i.e., one or more than one) elements $x \in X$ such that $y = f(x)$ then we say the mapping f is* ***surjective****. A surjective mapping is called a* ***surjection****.*

In other words, in a surjective mapping the range of f coincides with the codomain Y.

DEFINITEION 1.4.13. *Onto Mapping : This is another term for a* ***surjective*** *mapping.*

We say a surjection maps X **onto** the set Y, and if possible we write $f : X \xrightarrow{onto} Y$ or $f : X \xrightarrow{sur} Y$ or $f : X \twoheadrightarrow Y$. When a mapping is not known to be surjective we say f maps X **into** the Y, like this $f : X \xrightarrow{into} Y$.

Remark 1.4.4. We understand that the range of f is the image set $f[X]$. Hence in a surjective mapping, we have $Y = f[X]$.

Remark 1.4.5. Further it is always possible to restrict the codomain of a mapping such that the mapping f converts to a surjective mapping. This restriction is different with restriction of mapping in its domain. The surjective mapping then will be $f : X \longrightarrow f[X] \subseteq Y$. Still, we use the same notation f for our surjective mapping produced in this way.

This restriction later will be of some use in understanding topological embedding.
To show that a mapping is surjective we should take each y in the co-domain Y and check if there exists an x in domain X such that we can ensure that the selected y is the image of that x. We express this test in mathematical form as,

$$\forall y \in Y \exists x \in X \ni y = f(x)$$

DEFINITEION 1.4.14. *Saturated Sets : Assume $f : X \longrightarrow Y$ is a surjection and $A \subset X$. A is called a* ***saturated*** *subset of X* with respect *to f if A contains every subset $f^{-1}[y]$ that intersects with A.*

In another word, A is equal to inverse image $f^{-1}[B]$ of some subset $B \subset Y$.

Remark 1.4.6. If A is not saturated with respect to f then generally, $A \subset f^{-1}[f[A]]$. But for a saturated subset A we have $A = f^{-1}[f[A]]$.

Remark 1.4.7. Later we see that if f is a continuous map, a saturated set helps to define a **quotient** map and a quotient topology.

DEFINITEION 1.4.15. *Injective Mappings : Assume $f : X \longrightarrow Y$ is a mapping. If for each $y \in Y$ we can find* ***only*** *one (i.e.,* ***not*** *more than one) elements $x \in X$ such that $y = f(x)$ then we say the mapping f is* ***injective****. An injective mapping is called an* ***injection****.*

Remark 1.4.8. In an injective mapping $f : X \longrightarrow Y$ if $x_1 \neq x_2$ when $x_1, x_2 \in X$ then we have $f(x_1) \neq f(x_2)$. This is a way that you can check a mapping is an injection. See Figure 1.3

DEFINITEION 1.4.16. *One-one Mapping : Sometimes is termed as one-to-one mapping, this is another term for an* ***injective*** *mapping. We show an injection as $f : X \xrightarrow{inj} Y$ or $f : X \xrightarrow{1-1} Y$ or $f : X \rightarrowtail Y$.*

An idea similar to saturated sets can be explored to defining co-saturated sets.

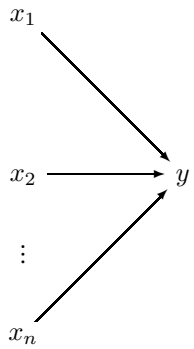


FIGURE 1.3. This can**not** happen in an injective mapping.

DEFINITEION 1.4.17. *Co-saturated Sets : Assume $f : X \longrightarrow Y$ is an injection and $B \subset Y$. B is called a **co-saturated** subset of Y* with respect *to f if B contains every subset $f[\{x\}] = \{f(x)\}$ that intersects with B where $x \in A$ and $A \subset X$.*

In another word, B is equal to image $f[A]$ of some subset $A \subset X$.

Remark 1.4.9. If B is not co-saturated with respect to f then generally, $ff^{-1}[B] \subset B$. But for a saturated subset B we have $ff^{-1}[B] = B$.

Remark 1.4.10. Later we see that if f is a continuous map, a co-saturated set helps to define an **induced** map and an induced topology.

Saturated and co-saturated sets help us to keep in mind properties of mappings defined on the intersection of sets with respect to the intersection of their images and also inverse image of intersections.

Remark 1.4.11. Assume $f : X \longrightarrow Y$ and $A \subset X$ and $B \subset X$ and $E \subset Y$ and $F \subset Y$ then, generally we have,

(1) $\quad if \quad A \subseteq B \quad then \quad f(A) \subseteq f(B)$
(2) $\quad f(A \cup B) = f(A) \cup f(B)$
(3) $\quad f(A \cap B) \subseteq f(A) \cap f(B)$
(4) $\quad f(A - B) \supseteq f(A) - f(B)$
(5) $\quad f(C(A)) \supseteq C(f(A))$
(6) $\quad f(A - B) \subseteq f(A)$
(7) $\quad if \quad E \subseteq F \quad then \quad f^{-1}(E) \subseteq f^{-1}(F)$
(8) $\quad f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$
(9) $\quad f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$
(10) $\quad f^{-1}(E - F) = f^{-1}(E) - f^{-1}(F)$
(11) $\quad f^{-1}(C(E)) = C(f^{-1}(E))$

Now we define a bijective mapping where all subsets of domain and co-domain are saturated and co-saturated respectively for the mapping.

DEFINITEION 1.4.18. *Bijective Mappings : A bijective mapping is one that is both a surjective mapping and an injective mapping. A bijective mapping is also is said*

to be a ***bijection****. We show an injection as* $f : X \xrightarrow{bi} Y$ *or as* $f : X \xrightarrow[onto]{1-1} Y$ *or* $f : X \leftrightarrow Y$.

Frequently a bijection might be referred as a **one to one correspondence**. One important bijection on a finite set is **permutation**.

DEFINITEION 1.4.19. *Permutation : A bijection* σ *on a finite set* S *is said to be a permutation on* S. *Hence,* $\sigma : S \xrightarrow{bi} S$.

DEFINITEION 1.4.20. *Operator : Operator is a mapping where domain is a Cartesian product of a set* $X \times X$ *and co-domain is the set* X *itself. An operator mapping is called an operation.*

DEFINITEION 1.4.21. *n−ary Operator : Is defined as a mapping from Cartesian product* X^n ***into*** *the set* X. *That is,* $f : X^n \longrightarrow X$.

Remark 1.4.12. Degenerate Case : It is worth noting the degenerate case of $n = 0$, where, you can remember, X^n defined to be $X^0 \triangleq \{(\,)\}$. In this case we assume $f(\{\varnothing\})$ is a member of X. This operator, $f : X^0 \longrightarrow X$ is called a **nulary** operator. An examples of a nulary operator is the 0 element and the 1 element in a Boolean algebra (Section 2.4).

Remark 1.4.13. Similarly, we have **unary** operator $f : X^1 \longrightarrow X$. An example of a unary operator is negation of an integer in additive group of integers, as later we study.

In a unary operation each element belonging to domain appears as a singleton, say, $\{x\}$ while the image of that element is just a member of the co-domain, i.e., $x \in X$.

DEFINITEION 1.4.22. *Pre-set of a Set : Assume* X *and* Y *are sets. Then the set of all mappings from* X *into* Y, *that is,* $\{\forall f \,|\, f : X \longrightarrow Y\}$ *is said to be the pre-set of set* X. *We show the pre-set by* ${}^{X}Y$ *notation.*

DEFINITEION 1.4.23. *f−dual Set of a Set : Assume* X *is a set. Then the set of all mappings from* X *into* X, *that is,* $\{\forall f \,|\, f : X \longrightarrow X\}$ *is said to be the f−dual set of the set* X. *We show this set by* X^{*f}.

This X^{*f} is the same as pre-set ${}^{X}X$.
We should study some degenerate cases for mappings before we could move further.

(1) Mapping $f : \varnothing \longrightarrow Y$. You can reason that this mapping is just the empty set $f = \varnothing$
(2) Mapping $f : \varnothing \longrightarrow \varnothing$. We can decide that this one is a special case of above and hence, $f = \varnothing$.
(3) Mapping $f : X \longrightarrow \varnothing$. This has no meaning. We do not define a mapping with **non-**empty domain and an empty co-domain. It means as if you have a mapping and then you do not assign anything to members of its domain. But mapping means assigning something to domain. Hence a contradiction.

(4) Pre-set ${}^{\varnothing}Y$. This has only one member. So, ${}^{\varnothing}Y = \{\varnothing\}$.
(5) Pre-set ${}^{\varnothing}\varnothing$. This is special case of above and ${}^{\varnothing}\varnothing = \{\varnothing\}$.
(6) Pre-set ${}^{X}\varnothing$. We might decide that this set is just an empty set, i.e., ${}^{X}\varnothing = \varnothing$.

DEFINITEION 1.4.24. *Power Set Mapping : Assume $f : X \longrightarrow Y$ is a mapping. We can define a mapping shown as 2^f from power set 2^Y to power set 2^X, that is, $2^f : 2^Y \longrightarrow 2^X$ such that if $S \in 2^Y$ then $2^f(S) = f^{-1}(S)$.*

DEFINITEION 1.4.25. *Restriction of a Mapping : Assume mapping $f : X \longrightarrow Y$ is defined. Then if we have a subset $A \subseteq X$ we can define the mapping $g : A \longrightarrow Y$ such that $g(x) = f(x), \forall x \in A$ as the restriction of f to the subset A. We show restriction of f to A by $f|_A \triangleq g$ notation.*

DEFINITEION 1.4.26. *Extension of a Mapping : Assume mapping $f : X \longrightarrow Y$ is defined. Then if we have a set $\Omega \supseteq X$ we can define the mapping $g : \Omega \longrightarrow Y$ such that $g(x) = f(x), \forall x \in X$ as the extension of f to the set Ω. We show extension of f to Ω by $f|^{\Omega}$ notation.*

Note that g needs to be consistent with f only on their common domain X.

DEFINITEION 1.4.27. *Composition of Mappings : Assume X and Y and Z are three sets and $W \subseteq Y$. Further, consider mappings $f : X \longrightarrow Y$ and $g : W \longrightarrow Z$. If the range $f[X]$ of f has common elements with domain W of g, that is, if $f[X] \cap W \neq \varnothing$ then we can define **composite** mapping of f and g shown as $g \circ f$ or simply gf by $gf : X \longrightarrow Z$.*

Hence f takes x to the y and then g takes y to the z, such that overall gf takes x to the z. To this end, g is restricted to $f[X] \cap W$ as its domain for composition.

DEFINITEION 1.4.28. *Submodulus Set : Assume mapping $f : X \longrightarrow X$ is defined. Then if $f[X] \subset X$, we say X is **submodulus set** of f.*

DEFINITEION 1.4.29. *Modulus Set : Assume mapping $f : X \longrightarrow X$ is defined. Then if $f[X] = X$, that is if f is a surjection, we say X is **modulus set** of f.*

DEFINITEION 1.4.30. *Identity Mapping : assume mapping $id : X \longrightarrow X$ is defined such that, for $x \in X$ we have $x \longmapsto x$, or $id(x) = x$ then id is called the identity mapping. We show the identity mapping on a set X by id_X.*

DEFINITEION 1.4.31. *Inclusion Mapping : Assume we have a subset $A \subseteq X$ then the restriction of the identity mapping $id : X \longrightarrow X$ to the subset A, that is $id_X|_A$ is said to be the inclusion mapping of A and is shown as ι_A.*

DEFINITEION 1.4.32. *Embedding Mapping : Assume we have a set $\Omega \supseteq X$ then the extension of the identity mapping $id : X \longrightarrow X$ to the set Ω, that is $id_X|^{\Omega}$ is said to be the embedding mapping of X into Ω. X is said as embedded in Ω.*

Frequently, embedding might be spelled as imbedding. It is better not to contrast this two different spelling as different concepts. Note that this 1.4.32 is one **basic** interpretation of embedding. This definition opens the way for later and further understanding of more sophisticated usage of the notion.

Remark 1.4.14. We notice that we can have two types of restriction and two types of extension for a mapping $f : X \longrightarrow Y$.

(1) Restriction of domain X.
(2) Extension of domain X.
(3) Restriction of co-domain Y.
(4) Extension of co-domain Y.

Often it is helpful to show composition of mappings in diagrams. It is generalization of arrow notation we already have used. For example, assume we have composition gf of g and f, with restriction we already imposed on the domain of g. We can show it by the diagram shown in Figure 1.4.

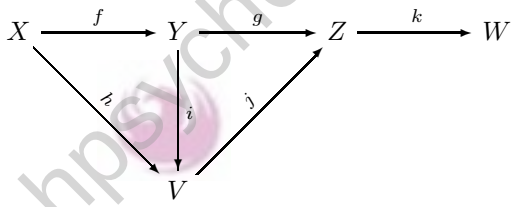


FIGURE 1.4. A Commutative Diagram.

You notice that from any domain to another co-domain there is one or more than one path consisting of one or many mappings. For example from X to V we have h or alternatively, through Y we have the composition if. From X to Z we have three paths gf and jh and jif.

DEFINITEION 1.4.33. *Commutative Diagrams : A commutative diagram shows equivalent composition of mappings on equivalent* ***paths****.*

AXIOM 1. *Axiom of Choice : For any relation R from a set X to another set Y there is a mapping $f : X \longrightarrow Y$.*

DEFINITEION 1.4.34. *Inverse Mapping : Assume $f : X \xrightarrow{inj} Y$. Then the mapping $g : f[X] \xrightarrow{inj} X$ is called* ***the*** *inverse mapping of mapping f.*

We show the inverse mapping of f with f^{-1}. You may notice that f is also inverse of the mapping $f^{-1} : f[X] \xrightarrow{inj} X$.

Remark 1.4.15. It can be easily observed that $f \circ f^{-1} = id_X$ and also $f^{-1} \circ f = id_{f[X]}$.

We can have the following diagram.

DEFINITEION 1.4.35. *Invertible Mapping : A bijective mapping is said to be an invertible mapping.*

DEFINITEION 1.4.36. *Gluing Mapping : Assume two mappings, $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ agree on the intersection $V = X \cap Y$ in the sense that $f(v) = g(v)$; $\forall v \in V$. Consider the mapping $\phi : X \cup Y \longrightarrow Z$ such that, $\phi(x) = f(x)$; $\forall x \in X$ and $\phi(y) = g(y)$; $\forall y \in Y$. Then we say that ϕ is the gluing mapping or better to say, ϕ is formed by gluing f and g, sometimes shown as $f \cup g$.*

DEFINITEION 1.4.37. *Iteration of Mappings : Assume $f : X \xrightarrow{inj} X$. Then X is a submodulus set with respect to f. We can define iterations of f by induction as,*

$$f^0(x) = x, \quad and \quad f^{n+1}(x) = f \circ f^n(x), \quad \forall x \in X \quad and \quad \forall n > 0.$$

If f is a bijection then X is a modulus set with respect to f and we can extend this definition to negative integers, as well, by defining,

$$f^{n-1}(x) = f^{-1} \circ f^n(x), \quad \forall x \in X \quad and \quad \forall n \leq 0.$$

Further we can define,

$$f^{n+m}(x) = f^m \circ f^n(x), \quad \forall x \in X \quad and \quad \forall n, m \in \mathbb{Z}.$$

Remark 1.4.16. Please remember following definition from basic calculus.

(1) Identity Map: is a map $f : X \longrightarrow X$ such that $x \mapsto x$ for all $x \in X$; that is, $f(x) = x$.
(2) Constant Map: is a map $f : X \longrightarrow Y$ such that there is a fixed $c \in Y$ that $x \mapsto c$ for all $x \in X$; that is $f(x) = c$.

DEFINITEION 1.4.38. *Function : A function is a mapping from any set X to the set of real numbers $\mathbb{R}$ or real $n-$tuples $\mathbb{R}^n$. That is $f : X \longrightarrow \mathbb{R}^n \quad \forall n \in \mathbb{N}$.*

DEFINITEION 1.4.39. *Trivial Function : Is a constant mapping function $f : X \longrightarrow \mathbb{R}$ such that $f(x) = 0$.*

In later contexts of group theory or vector spaces this might be called trivial mapping or trivial functional.

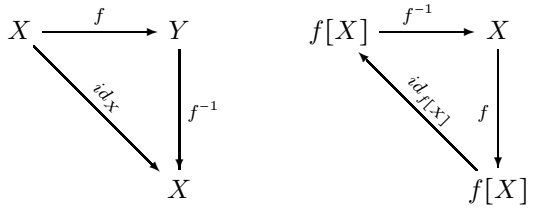


FIGURE 1.5. Commutative Diagrams for Inverse Mappings.

Definiteion 1.4.40. *Kernel of a Function : Kernel is the set of all element in X whose images in $\mathbb{R}$ is the single element zero $0 \in \mathbb{R}$. In other words, kernel is the pre-image $f^{-1}[\{0\}]$ of singleton $\{0\} \subseteq \mathbb{R}^n$. See, Figure 1.6*

Later we define kernel for mappings into the certain other algebraic structures besides $\mathbb{R}$ when there is a neutral element 0 defined on them.

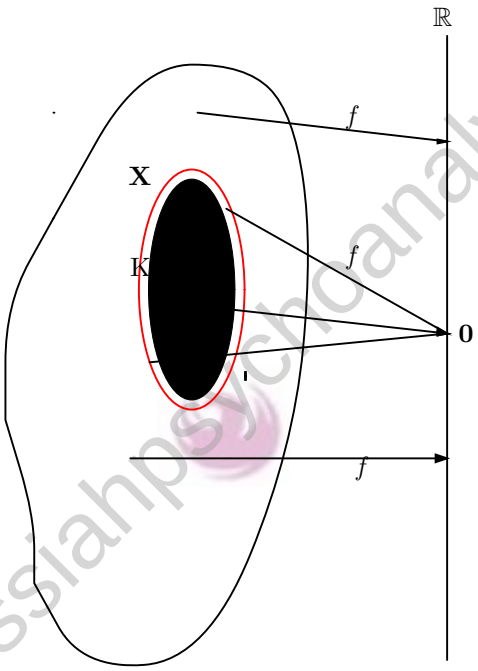


Figure 1.6. Kernel of function f is everything on red boundary and inside it.

Example 1.4.3. *Forms: Forms are examples of functions on a tangent space. They act on a vector in the tangent space and give a real number attributed or attached to that vector.*

Example 1.4.4. *Co-vectors: Corresponding to a vector in a tangent space, it is possible to make a form from components of that vector. This form is called a co-vector or a row vector, in contrast to a column vector. A co-vector, like a form, is a function.*

Definiteion 1.4.41. *Product of Mappings: Let $f : X \longrightarrow V$ and $g : Y \longrightarrow W$ be two mappings. Then we can define a mapping $f \times g : X \times Y \longrightarrow V \times W$ as the product of these two mappings in such a way that,*

$$(f \times g)(x, y) = (f(x), g(y)) = (u, v) \quad \text{where} \quad u = f(x), \quad \text{and} \quad v = g(y).$$

Figure, 1.7

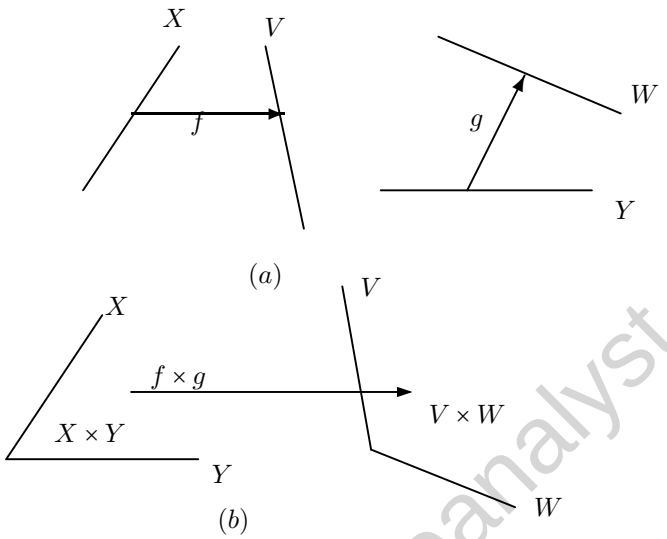


FIGURE 1.7. Product of two maps.

You might notice that we were quite terse on this definition. Well, one should be cautious on using this definition. But delving on it will ease the way for understanding other complicated ideas such as the product of measures, bilinear mapping, tensor product, and n–forms, (topological) quotient map, cotangent space, and "pull-back." In particular, there are situations where one decides that could be a two variables mapping, instead of product of two mappings.
In the same way that a mapping from a set X to the set $\mathbb{R}$ has its own name as a function, a map from a subset of $\mathbb{R}$ to a set X has a wide usage as a **path** mapping. We have other types of combination of mappings such as $f \vee g$ for **join** of the mappings and $f \wedge g$ for the **meet** of the mappings and objects such as f^+ and f^- that will be discussed on their own places, if necessary.

DEFINITEION 1.4.42. *Path Mapping : A mapping $f : [a, b] \longrightarrow X$ is called a path mapping for $[a, b] \subset \mathbb{R}$.*

This is the first encounter with the idea of path. Actually a path needs to have more restrictions to be well-defined and has its own context.

CHAPTER 2

Structures on sets

2.1. Relations

Assume $\mathcal{R}$ is a relation on a set X. Then the relation is called a,

DEFINITEION 2.1.1. *Reflexive Relation : If and only if for each $x \in X$ we have $x\mathcal{R}x$.*

An example of such relation is less than or equal relation ($\leq$) on the set of , say integers. For any integer x we have $x \leq x$.

DEFINITEION 2.1.2. *Irreflexive Relation : If and only if there is **not** any $x \in X$ such that $x\mathcal{R}x$ holds.*

For example the relation less than ($<$) on the set of integers $\mathbb{Z}$ does **not** hold. For any integer x it is **not** true that $x < x$.

DEFINITEION 2.1.3. *Symmetric Relation : If and only if $x\mathcal{R}y$ implies $y\mathcal{R}x$ for all $x, y \in X$.*

An example is the equality relation $x = y$

DEFINITEION 2.1.4. *Anti-symmetric Relation: If and only if $x\mathcal{R}y$ does **not** imply $y\mathcal{R}x$ but if $x\mathcal{R}y$ and $y\mathcal{R}x$ **both** hold then $x = y$.*

A familiar example of anti-symmetric relation is the less than ($<$) relation $x < y$.

DEFINITEION 2.1.5. *Transitive Relation: If and only if two relations $x\mathcal{R}y$ and $y\mathcal{R}z$ imply $x\mathcal{R}z$.*

Again the less than ($<$) relation $x < y$ satisfies transitivity requirement as a relation.

DEFINITEION 2.1.6. *A-transitive Relation: If and only if $x\mathcal{R}y$ and $y\mathcal{R}z$ imply $x\not\mathcal{R}z$, where slashed $\not\mathcal{R}$ means no such a relation established.*

Definiteion 2.1.7. *Trichotomy Relation : If and only if* ***only one*** *of the three relations $x\mathcal{R}y$, $y\mathcal{R}x$ or $x = y$ holds.*

Definiteion 2.1.8. *Equivalence Relation on X: This relation is reflexive, symmetric and transitive.*

Example 2.1.1. *Degenerated Case: $\varnothing$ is an equivalence relation on $\varnothing$.*

Definiteion 2.1.9. *Order Relation on X: This relation is reflexive, antisymmetric and transitive.*

Example 2.1.2. *Degenerated Case: $\varnothing$ is an order relation on $\varnothing$.*

Definiteion 2.1.10. *Strict Order Relation on X: This relation is irreflexive, anti-symmetric and transitive.*

Example 2.1.3. *Degenerated Case: $\varnothing$ is a strict order relation on $\varnothing$.*

2.2. Order in Sets

The order relation on sets is shown by the usual notation $\leq$ instead of $\mathcal{R}$.

Definiteion 2.2.1. *Dominant Set : Assume we have sets X and Y. We say Y is dominant over X, if there exists an injection f from X into the Y. That is, $f : X \xrightarrow{inj} Y$. We show this relation by $X \preccurlyeq Y$.*

If Y is dominant over X then we have $f[X] \subseteq Y$ and $card(X) \leq card(Y)$.

Definiteion 2.2.2. *Preordered Sets : A set is preordered when there is a reflexive and transitive relation on some of the elements in the set.*

Definiteion 2.2.3. *Partially Ordered Sets : A set is partially ordered when there is a reflexive and* ***anti****symmetric, and transitive relation on* ***some*** *of the elements in the set.*

Definiteion 2.2.4. *Totally (or Linearly) Ordered Sets : A set is totally ordered when there is a reflexive and* ***anti****symmetric, and transitive relation on* ***all*** *of the elements in the set.*

Example 2.2.1. *Natural Numbers : Subset relation in sets is a reflexive and* ***anti****symmetric, and transitive relation on* ***all*** *the subsets of a particular set. From the section 1.3.1 we can remember that there is a subset relation for any pair of natural numbers. For example, $5 \subseteq 17$, that creates a natural order among elements of $\mathbb{N}$. We can see that all three necessary conditions are satisfied by this relation. We show this order relation by $\leq$; hence, $5 \leq 17$.*

Definiteion 2.2.5. *Chain : Any totally ordered subset of a partially ordered set is a chain.*

DEFINITEION 2.2.6. *Notation : Assume we have ordered set X ordered with relation R, we show it by notation $\mathfrak{O}(X,\ R)$.*

DEFINITEION 2.2.7. *Similarity : Assume $\mathfrak{O}(X,R)$ and $\mathfrak{O}(Y,S)$ are ordered sets. The mapping $f : X \longrightarrow Y$ is said to be a* ***similarity*** *if from the relation $x_1 R x_2$ $\forall x_1, x_2$ in domain X, we have $f(x_1)\, S f(x_1)$.*

DEFINITEION 2.2.8. *Order Preserving Mapping : Let $f : X \longrightarrow Y$ and $\forall x, \xi \in X$, $x \leq \xi$ then $f(x) \leq f(\xi)$.*

You notice that order preserving mapping is a similarity. This will be of use when we study notion of **embedding** in context of universal algebra.

2.3. Lattice and Well Ordering

Assume A is an ordered subset of an ordered set Ω. An element x in Ω is a lower bound for elements of A whenever for all elements a in A we have $x \leq a$. In mathematical notation we write

DEFINITEION 2.3.1. *Lower Bound : Let $A \subseteq \Omega$, then an element $x \in \Omega$ is a lower bound for A whenever $\forall\ a \in A$ we have $x \leq a$.*

Assume A is an ordered subset of an ordered set Ω. An element x in Ω is an upper bound for elements of A whenever for all elements a in A we have $a \leq x$. In mathematical notation we write

DEFINITEION 2.3.2. *Upper Bound : Let $A \subseteq \Omega$, then an element $x \in \Omega$ is an upper bound for A whenever $\forall\ a \in A$ we have $a \leq x$.*

Assume x and y are two elements belong to an ordered set Ω. We say y covers x if $x \leq y$, and additionally you cannot find a z that comes between x and y in their order. In mathematical notation we write

DEFINITEION 2.3.3. *Cover : Let Ω be a partially ordered set and $x,\ y \in \Omega$. We say y covers x whenever $x \leq y$ and there does not exist z such that $x \leq z \leq y$.*

DEFINITEION 2.3.4. *Least Upper Bound (Supremum): Let $A \subseteq \Omega$, then an element $x \in \Omega$ is the least upper bound for A whenever x is an upper bound for$A$ and if y is another upper bound for A then $x \leq y$.*

DEFINITEION 2.3.5. *Greatest Lower Bound (Infimum): Let $A \subseteq \Omega$, then an element $x \in \Omega$ is the greatest lower bound for A whenever x is a lower bound for$A$ and if y is another lower bound for A then $y \leq x$.*

DEFINITEION 2.3.6. *Maximum (Greatest) element : Let $A \subseteq \Omega$; further, if $x \in \Omega$ is the least upper bound of A then $x \in A$.*

The following two definitions establishes an order on a two elements subset.

DEFINITEION 2.3.7. *Minimum (Least) element : Let $A \subseteq \Omega$; further, if $x \in \Omega$ is the greatest lower bound of A then $x \in A$.*

DEFINITEION 2.3.8. *Join : Least upper bound of a set with two members a and b is called join of a and b we show it by $a \vee b$.*

DEFINITEION 2.3.9. *Meet : Greatest lower bound of a set with two members a and b is called join of a and b we show it by $a \wedge b$.*

DEFINITEION 2.3.10. *Directed Partially Ordered Sets: This is a set, where every pair of elements have an upper bound.*

DEFINITEION 2.3.11. *Well-ordered Sets : This is a set, where every non-empty subset has a minimum (least) element.*

Example 2.3.1. *Set of Natural Numbers : This set is a well ordered set.*

DEFINITEION 2.3.12. *Lattice : A lattice $\mathcal{L}$ is a partially ordered set such that,*

(1) *Every pair of elements of $\mathcal{L}$ have a joint and a meet (informally, every two elements are comparable as if they are next to each other).*
(2) *There are element $0 \in \mathcal{L}$ and element $1 \in \mathcal{L}$ such that for every element $a \in \mathcal{L}$ we have $0 \leq a \leq 1$*

DEFINITEION 2.3.13. *Complement : Assume $\mathcal{L}$ is a lattice, and $a \in \mathcal{L}$. Now, consider you can find $a' \in \mathcal{L}$ such that it satisfies the following conditions:*

(1) *$a' \vee a = 1$ (informally, the bigger is 1).*
(2) *$a' \wedge a = 0$ (informally, the smaller is 0).*

Then we say a' is ***the complement*** *of a. We use symbol $a^{'}$ to show complement of element $a \in \mathcal{L}$.*

It is trivial to see (by swapping the place of a and a' in above) that if a' is the complement of a, then a is also complement of a'; that is, $(a^{'})^{'} = a$

DEFINITEION 2.3.14. *Orthocomplemention Mapping: This is defined as mapping ω on lattice $\mathcal{L}$ as $\omega : \mathcal{L} \longrightarrow \mathcal{L}$ such that for any $a \in \mathcal{L}$ we have $a \mapsto a^{'}$, that is, it maps element $a \in \mathcal{L}$ to its complement $a^{'}$ Further,*

(1) *$(a^{'})^{'} = a$.*
(2) *if $a \leq b$ then we have $b^{'} \leq a^{'}$.*

DEFINITEION 2.3.15. *Orthocomplemented Lattice : A lattice with an orthocomplemented mapping defined in it is called orthocomplemented*

Definiteion 2.3.16. *Complete Lattice : When every non-empty subset of a lattice have a least upper bound and a greatest lower bound, the lattice is complete.*

Definiteion 2.3.17. *Distributive Lattice : regarding join and meet element for comparing three elements of a lattice L*

(1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

2.4. Boolean Algebra

Definiteion 2.4.1. *Boolean Algebra : An orthocomplemented, distributive lattice is called a Boolean algebra.*

Definiteion 2.4.2. *Infinite Distributive :*

(1) $a \bigwedge S = \bigwedge \{a \vee b \mid b \in S\}$
(2) $a \bigvee S = \bigvee \{a \wedge b \mid b \in S\}$

Definiteion 2.4.3. *Locale/Frame : Is a lattice with infinite distributive property.*

2.5. Partition

Assume Ω is a set and $\mathfrak{M}$ is a collection of subsets of Ω.

Definiteion 2.5.1. *Filter or Up-set : $\mathfrak{M}$ is a filter in Ω if for any $A \in \mathfrak{M}$ and $A \supseteq B$ then $B \in \mathfrak{M}$*

Definiteion 2.5.2. *Ideal or Down-set : $\mathfrak{M}$ is an ideal in Ω if for any $A \in \mathfrak{M}$ and $A \subseteq B$ then $B \in \mathfrak{M}$*

Definiteion 2.5.3. *Base $\mathfrak{B}$: $\mathfrak{B}$ is a base for $\mathfrak{M}$ if for any $A \in \mathfrak{M}$ there is $B \in \mathfrak{B}$ such that $B \subseteq A$*

Definiteion 2.5.4. *Refined Family : A collection $\mathfrak{N}$ is said to refine $\mathfrak{M}$ or to be a refinement of $\mathfrak{M}$ if for every $N \in \mathfrak{N}$ we have an $M \in \mathfrak{M}$ such that $N \subseteq M$.*

Definiteion 2.5.5. *Complete Family : For any $A \in \mathfrak{M}$ and $B \in \mathfrak{M}$ then $A \cap B \in \mathfrak{M}$*

Definiteion 2.5.6. *Anti-chain : For any $A_i \in \mathfrak{M}$ and $A_j \in \mathfrak{M}$ then $A_j \nsubseteq A_i$*

Definiteion 2.5.7. *Chain : For any $A_i \in \mathfrak{M}$ and $A_j \in \mathfrak{M}$ then either $A_i \subseteq A_j$ or $A_j \subseteq A_i$ is correct.*

DEFINITEION 2.5.8. *Maximal Chain : $\mathscr{C}$ is a maximal chain if for any other chain $\mathscr{C}'$ if $\mathscr{C} \subseteq \mathscr{C}'$ then $\mathscr{C} = \mathscr{C}'$*

DEFINITEION 2.5.9. *Partition (Finite): The class of sets $\{ A_j \}$ such that $A_i \subseteq \Omega$; $i = 1, 2, \cdots, n$ and $A_i \cap A_j = \varnothing$ when $i \neq j$ and $\Omega = \bigcup_{i=1}^{n} A_i$.*

DEFINITEION 2.5.10. *Partition (Countable): The class of sets $\{ A_j \}$ such that $A_i \subseteq \Omega$; $i = 1, 2, \cdots$ and $A_i \cap A_j = \varnothing$ when $i \neq j$ and $\Omega = \bigcup_{i=1}^{\infty} A_i$.*

DEFINITEION 2.5.11. *$\mathfrak{M}$–partition : The class of subsets $\{A_i\}$ such that*

(1) *each $A_i \in \mathfrak{M}$.*
(2) *$\{A_i\}$ is a partition.*

DEFINITEION 2.5.12. *Dissection : Dissection is the less common word for partition.*

Remark 2.5.1. Important (Repeat Remark 1.2.1): We know what is a power set. Frequently we need to select certain collections of subsets of a set with certain structure out of the entire collection of subsets. For example $\mathfrak{M}$ which is a subcollection of $\mathscr{P}$. That is, $\mathfrak{M} \subseteq \mathscr{P}$. When we freely select an arbitrary collection and like to impose certain structure to them we call that collection a free collection and we show it by $\mathscr{F}$. To impose the certain structure to this collection $\mathscr{F}$, we make an intersection over all those collections that have that structure and contain $\mathscr{F}$ as a subset. Then we have the **smallest** collection shown say by $\mathscr{F}^*$ that is endowed with our desired structure. We easily can verify that having any two sets in $\mathscr{F}^*$ then we have their intersection in $\mathscr{F}^*$. Also if a set belongs to $\mathscr{F}^*$ then all of its subsets also belong to $\mathscr{F}^*$.

AXIOM 2. *Axiom of Choice: Every partially ordered set has a maximal chain.*

2.6. Quotient Set

DEFINITEION 2.6.1. *Equivalence Set (Class) : Assume R is an equivalence relation on set X. Choose an element $\xi \in X$. We show the set of all elements $x \in X$ that are related to ξ, that is, ξRx by symbol $[\, \xi \,]_R$ and call it equivalence set (class) of element ξ, conveniently written as $[\, \xi \,]$.*

Assume we have a box of screws in different diameters, and we like to separate them according to their diameters. The best way is to take a nut, test each screw that fits that nut and put all such a screws in a pack. Glue the nut on the pack for future reference or if a screw found somewhere not tested yet and needed to be tested. You get few packs of **class**ified bolts in place of one box of mixed screws. Each pack is represented with a nut instead of bolts. We say all bolts in the pack are congruent modulo that nut. We do not have screws any more. We have packs. We partitioned original box of bolts into the equivalent classes. You can identify all bolts inside each pack with the nut glued on the packs. We know nuts are different from bolts. Sometimes you might decide for the partitioning of the screws inside

each pack, say, based on the length of bolts.
Frequently we use the words identify and identification. By that we mean, "consider or treat many objects as the same or associate all of them to the same thing." In the same way that in a haberdasher's shop each button attached on a box leads to the idea that all buttons in that box are the same as that one. You cannot and won't recognise them from each other.

DEFINITEION 2.6.2. *Identification : When we have an equivalence set* $[\,\xi\,]$ *for a* $\xi \in X$*, we say this set is an* ***identification*** *of all elements* $x \in X$ *related to* ξ *through* R*. All elements* $x \in X$ *are identified by set* $[\,\xi\,]_R$*.*

Note that when you identify a set of elements $x \in X$ by $[\,\xi\,]_R$, then you have exhausted all those elements to the single class $[\,\xi\,]_R$. $[\,\xi\,]_R$ is a box and ξ is the representative (the button) on the box. Each $[\,\xi\,]_R$ is one element of a superset, say Ω, which is different from the set X. X is not a subset of Ω and Ω is not a subset of X.

DEFINITEION 2.6.3. *Identification, Division (Quotient) of a Set by its Subset : Assume* X *is a non-empty set and* $A \subset X$ *and* $V = X - A$*. Consider we use the monstrous meticulous notation* $\bigcup V \cup \{A\}$*. We show this set as* X/A*, as all the elements of subset* A *are shrunk to (identified by) one of its elements and put together with the remaining elements of* X *in a superset shown as* X/A*. Please differentiate it with the complement set* $X \backslash A$*.*

The common, identifying, character of the boxed elements is merely their belongness to the subset A. We identify all elements of A as one and label them by A.
Hence we make a set consisting of elements of $X - A$, and we put whole set A as one point next to those elements as the representative of all the removed elements. Assume we have a box of loose balls with different colours. We can find four red ball among all the balls. We separate them in a small box A and put this small box back in the original box among other balls. Now with one look we can identify the box of the all red balls among all the balls.

Example 2.6.1. *Cone : Assume* $I = [0, 1] \subset \mathbb{R}$*. and* X *is any set. The Cartesian product* $X \times I$ *is called a solid Cylinder of unit height. A subset of this set is the set* $X \times \{1\}$*, which constitute its top cross section. Then the quotient* $(X \times I)/(X \times \{1\})$ *is called a cone with vertex at point 1. Hence, a cone is the quotient (division) of a cylinder by its top surface. I put parenthesis for clarity. I do not need them. Hence,* $X \times I/X \times \{1\}$*.*

EXERCISE 6. *An easy exercise : If* $\xi \neq \zeta$*, then either* $[\,\xi\,]_R \cap [\,\zeta\,]_R = \varnothing$ *or* $[\,\xi\,]_R = [\,\zeta\,]_R$*.*

Remember those boxes of the bolts.

DEFINITEION 2.6.4. *Identifying Map (Kuratowski) : Assume* f *is a mapping of* X *to* Y*, that is* $f : X \longrightarrow Y$*. Take* $\xi \in X$ *and define relation* R *on* X *as* $xR\xi$ *for*

*$x \in X$ if and only if $f(x) = f(\xi)$. We say that the mapping f **identifies** a class of elements of X. (Also please remember the kernel of a function 1.4.40)*

It is easy to prove that this relation is an equivalence relation on X. Given $\xi \in X$, the set of all $x \in X$ equivalent modulo f to ξ, is shown as, $[\xi]_f$, and sometimes is referred as the orbit of ξ under f.

Remark 2.6.1. Kuratowsky identification actually is defined when we define relation R on X as $xR\xi$ for $x \in X$ if and only if $f^m(x) = f^n(\xi)$ for $\xi \in X$ and some $n, m \in \mathbb{Z}$, where X is a modulus set with respect to mapping f. Identification of ξ is shown as $[\xi]_f^{m,n}$, and reads as orbit of ξ with respect to f **of order** m, n.

Example 2.6.2. *: Let's define $b : \mathbb{Z} \longrightarrow \{0,1\}$ such that $b(z) = 0$ whenever z is even, and otherwise, $b(z) = 1$ when z is odd. Hence even integers are identified by 0 through the mapping b and odd integers by 1. Then, $\mathbb{Z}/b = \{[3]_b, [4]_b\}$.*

Definiteion 2.6.5. *Fixed Point : Take ξ in X and assume X is a submodulus set with respect to the mapping $f : X \longrightarrow X$. Then $[\xi]_f^{0,1}$, that is the set of all $\xi = f^k(\xi)$ are called fixed points of f of order k.*

Definiteion 2.6.6. *Quotient Set X/R : Assume R is an equivalence relation in X. For each $\xi \in X$, there is an equivalence set $[\,\xi\,]_R$. Then the set $\{[\,\xi\,]_R | \forall \xi \in X\}$ is called the **quotient set** of X **modulo** R, shown as X/R.*

- Each equivalence set $[\,\xi\,]_R$ exhausts certain subset of X to a single class. The set of all these classes is the quotient set X/R.
- Quotient set X/R makes a partition on set X.
- In the Example 2.6.2 above we showed the quotient set as $\mathbb{Z}/b$.

Definiteion 2.6.7. *Canonical map π : The map $\pi : X \longrightarrow X/R$, where $\xi \mapsto [\,\xi\,]_R$ is called canonical map on X. Canonical map sometimes is referred as the natural map.*

The canonical map π is surjective.

Example 2.6.3. *Nuts and Bolts: Remember the box of bolts already discussed. Assume a mapping (actually a function) $f : B \to \mathbb{R}^+$ from the box of loose bolts B to positive real numbers assigns to each bolt its diameter in millimeter. There is a canonical map $\pi : B \longrightarrow B/\sim$ from the loose box of all bolts to the partitioned collection of packs of equal diameter bolts that assigns one nut for each pack. Now there exists a unique injective function $\widehat{f} : B/\sim \longrightarrow \mathbb{R}^+$ such that $f = \widehat{f} \circ \pi$. See Figure 2.1.*

Remark 2.6.2. In reference to definition 2.6.4, we can define a mapping $\kappa : X/f \longrightarrow f(Y)$ (this is a bijection). This should be contrasted clearly with the canonical mapping, 2.6.7, defined above.

Example 2.6.4. *Let $0 \le \xi \le 1$. Identify all $x \in \mathbb{R}$ by relation $\xi R x$, when, $x = k + \xi$, for all integers $k \in \mathbb{Z}$. Quotient set X/R is the unit circle $\mathbb{S}^1$, or the unit circle in the complex plane. Each $[\,\xi\,]_R$ is one point on the circle and each addition of the integer k rotates once round the circle.*

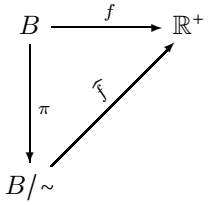


FIGURE 2.1. Commutative Diagram for Partitioning Nuts and Bolts.

Please differentiate the previous identification with $\mathbb{R}/\mathbb{Z}$, where one pinches all points in $\mathbb{Z}$ as one point in $0 \in \mathbb{R}$. Result is a bouquet of infinite circles at the origin. Take a tape ($\mathbb{R}$) punched in equal intervals ($\mathbb{Z}$) like a waist belt. Pass a String through the holes. Then pull the string tight together and fasten ends of the string. You are going to have a bouquet of circles attached at the holes ($\mathbb{R}/\mathbb{Z}$). Previous example will be visited again (4.2.1) in shade of the quotient of additive group $\mathbb{R}$ with respect to cosets of the subgroup $\mathbb{Z}$; that is, $\mathbb{R}/\mathbb{Z}$. Therefore we have two different $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}$.
Perhaps you can remember the Gluing Mapping, from the definition 1.4.36. There is another idea near to it we define as

DEFINITEION 2.6.8. *Attaching Map : Assume we have sets X and Y and the subset $A \subseteq X$ and the mapping $f : A \longrightarrow Y$. Pinch each $a \in A$ to its image $f(a) \in Y$. The set of all points resulted in this way together with the points in $X - A$ and points in the $Y - f(A)$ as they are, are said to be attached by the attaching map f. The resulted identified set is shown as $X \cup_f Y$*

Assume we have a box of nuts and a box of bolts. Define the mapping f as one that assigns a bolt of correct size from the second box (Y) to each nut from the first box (X). These proper nuts form a subset (A) of all nuts. Separate pairs of fitted nuts and bolts and fix them together and return them with the left-alone nuts and bolts in a joint box. Now we have a common box of separate nuts without matched bolts and bolts without matched nuts and some attached nuts and bolts. The mentioned f is an attaching map and the newly arranged box is the partition of the union of the two boxes. Had we had a box of fifty nuts and a box of fifty bolts and could find ten matched bolts and ten matched nuts (we could have more than ten nuts since we did not emphasise f to be an injection, fitting more than one nut around a bolt) the newly formed set has got 80 elements or less instead of 100. This is a partitioning of the original mixture of nuts and bolts. Since we have gathered those elements that can be related by our proposed mapping f, and reduced the size of the set as desired. Also this does not show congruence among the fitted nuts and bolts themselves more than it should. Since, for instance, we only considered fitting of one or more nuts to a bolt not that we have separated all bolts with the similar thickness or length.
Please note that to create the attaching map first we build a **disjoint union** of nuts and bolts in a common box. To do that we should label contents of the first box with something recognizable, say its box number, 1. And label all the content of the second box with another number, say, 2. For these elements we did not need such a labeling since the elements are physically recognizable by their shapes. We

have a box of hardware; some labeled as nuts and some labeled as bolts. If the sets were abstract sets such as the real numbers or sets of points in an Euclidean plane then such a labeling for mixing them together would be imperative.
This is a preparation for later concise definition of an attaching map that requires continuity of f and topological spaces X and Y. In discussing degenerate cases of empty sets, we succeeded to build all whole numbers from zero to any arbitrary large number. We know natural numbers are natural in the sense that human started counting with them. This exclude zero from the set of natural numbers. Later human discovered zero and after that negative number. Having natural numbers in hand, how can we build zero and negative integers from them without additional material? The only thing we need is definition of Cartesian product, and from it the definition of an equivalence relation. We show the set of positive (natural) numbers by $\mathbb{N}^\dagger$

Example 2.6.5. *Set of Integers $\mathbb{Z}$: Take $(\mu,\nu) \in \mathbb{N}^\dagger \times \mathbb{N}^\dagger$ and define relation shown with symbol $\sim$ by identifying all $(m,n) \in \mathbb{N}^\dagger \times \mathbb{N}^\dagger$ as $(m,n) \sim (\mu,\nu)$ if and only if $m+\nu = n+\mu$. Then the equivalence set (class) $[(\mu,\nu)]$ is said to be an integer. The set of all integers is shown by $\mathbb{Z}$. Note that $0 \triangleq [(\mu,\mu)]$.*

Example 2.6.6. *Order in the Set of Integers $\mathbb{Z}$: Remember we created a natural order in the set of natural numbers by the $\subseteq$ relation. Take $(\mu,\nu) \in \mathbb{N}^\dagger \times \mathbb{N}^\dagger$ and define relation shown with symbol $\leq$ by all $[(m,n)]$, $[(\mu,\nu)] \in \mathbb{Z}$ if and only if $m+\nu \leq n+\mu$. Then this relation is reflexive, antisymmetric, and transitive in ordering the equivalent classes of the form $[(m,n)]$; in the sense that, if $z_1 = [(m_1,n_1)]$ and $z_2 = [(m_2,n_2)]$. Then $z_1 \leq z_2$ if and only if $m_1+n_2 \leq m_2+n_1$.*

Definiteion 2.6.9. *Congruence Relation : Take two integers a and b. If their difference is an integer multiple of some non-zero integer k, that is, $a-b = n.k$, we say a and b are related **modulo** n and we show it by $a \equiv b\,(modulo\ n)$. We read this as a is **congruent** to b modulo n.*

In another word, we know two congruent integers as equal, or rather equivalent numbers. Hence all congruent numbers (**modulo** n) exhaust or are identified by a single class $[\quad]_{\equiv(modulo\ n)}$.

Definiteion 2.6.10. *$\mathbb{Z}_n$: Take $\zeta \in \mathbb{Z}$, the set of all $z \in \mathbb{Z}$ such that $z \equiv \zeta\,(modulo\ n)$ (that is, $z-\zeta = k.n$) is the equivalent class $[\zeta]_{\equiv(modulo\ n)}$.*
We define $\mathbb{Z}_n = \left\{ [\zeta]_{\equiv(modulo\ n)} \,|\, \zeta \in \mathbb{Z} \right\}$

After now we show $[\zeta]_{\equiv(modulo\ n)}$ simply as $[\zeta]_n$.

Example 2.6.7. *$\mathbb{Z}_1$: For each $\zeta \in \mathbb{Z}$ we should take all $z \in \mathbb{Z}$ such that $\zeta \equiv z\,(modulo\ 1)$ (that is, $\zeta - z = k.1$).*
Assume $\zeta = 2$ and we like to construct $[\,2\,]_{\equiv(modulo1)}$, or $[\,2\,]_1$. Then the set of all $z \in \mathbb{Z}$ such that $z-2 = k$, or $z = 2+k$, $\forall k \in \mathbb{Z}$ coincides with the set of integers $\mathbb{Z}$. Hence, $[\,2\,]_1 = \{\cdots,-3,-2,-1,0,+1,+2,+3,\cdots\} = \mathbb{Z}$. The same results for any $\zeta \in \mathbb{Z}$ other than 2. Hence, $[\,0\,]_1 = [\,1\,]_1 = [\,2\,]_1 = \cdots$. Then we have, $\mathbb{Z}_1 = \{[\,0\,]_1\}$.

Please note to distinguish between $\mathbb{Z}$ and $\mathbb{Z}_1$. Set $\mathbb{Z}$ has countably infinite members, while $\mathbb{Z}_1$ is a singleton set, has only **one** element $[\,0\,]_1$. Incidentally, we observe that $[\,0\,]_1 = \mathbb{Z}$ and also $\mathbb{Z}_1 = \{\mathbb{Z}\}$, and $\mathbb{Z}_1 \neq \mathbb{Z}$.

Example 2.6.8. $\mathbb{Z}_2$ *: For each* $\zeta \in \mathbb{Z}$ *we should take all* $z \in \mathbb{Z}$ *such that* $\zeta \equiv z\,(modulo\ 2)$ *(that is,* $\zeta - z = k.2$*).*
Assume $\zeta = 5$ *and we like to construct* $[\,5\,]_{\equiv(modulo2)}$*, or* $[\,5\,]_2$*. Then the set of all* $z \in \mathbb{Z}$ *such that* $z-5 = k.2$*, or* $z = 5+k.2 = 1+2.(2+k) = 1+2.m$*,* $\forall k \in \mathbb{Z}$ *coincides with the set of all odd integers. Hence,* $[\,5\,]_2 = \{\cdots,-7,-5,-3,-1,+1,+3,+5,+7\cdots\}$*. The same results for any other odd integer* $\zeta \in \mathbb{Z}$ *other than 5 such as 1 or 9 or 23. On the other hand, if we select an even integer, say,* $\zeta = 6$ *and construct* $[\,6\,]_{\equiv(modulo2)}$*, or* $[\,6\,]_2$*. Then the set of all* $z \in \mathbb{Z}$ *such that* $z-6 = k.2$*, or* $z = 6+k.2 = 2.(3+k)$*,* $\forall k \in \mathbb{Z}$ *will be the set of all even integers (since it is a multiple of 2). Hence,* $[\,6\,]_2 = \{\cdots,-8,-6,-4,-2,0,+2,+4,+6,+8\cdots\}$*. The same results for any other even integer* $\zeta \in \mathbb{Z}$ *other than 6 such as 2 or 10 or 18. Therefore,* $[\,0\,]_2 = [\,2\,]_2 = [\,4\,]_2 = \cdots$ *and* $[\,1\,]_2 = [\,3\,]_2 = [\,5\,]_2 = \cdots$*. Then we have,* $\mathbb{Z}_2 = \{[\,0\,]_2, [\,1\,]_2\}$*.*

$\mathbb{Z}_2$ has **two** members. All even integers are identified with one class (pinched as if they are all one point) $[\,0\,]_2$ and all odd integers are identified with another class $[\,1\,]_2$.

Example 2.6.9. $\mathbb{Z}_3$ *: For each* $\zeta \in \mathbb{Z}$ *we should take all* $z \in \mathbb{Z}$ *such that* $\zeta \equiv z\,(modulo\ 3)$ *(that is,* $\zeta - z = k.3$*).*
Assume $\zeta = 4$ *and we construct* $[\,4\,]_{\equiv(modulo3)}$*, or* $[\,4\,]_3$*. Then the set of all* $z \in \mathbb{Z}$ *such that* $z - 4 = k.3$*, or* $z = 4 + k.3$*,* $\forall k \in \mathbb{Z}$ *coincides with the set of all integers in form of,* $[\,3\,]_3 = \{\cdots,-12,-9,-6,-3,0,+3,+6,+9,+12\cdots\} = \{0+3.k|\forall k \in \mathbb{Z}\}$*. If we select any integer* ζ *equal to one of these integers the equivalence set would be equal to that again. We select* $[\,0\,]_3$ *as representative of this class.Now select an integer* $\zeta = 1+3.k$*, say,* $\zeta = 4$ *and construct* $[\,4\,]_3$*. Then the set of all* $z \in \mathbb{Z}$ *such that* $z-4 = k.3$*, or* $z = 4+k.3$*,* $\forall k \in \mathbb{Z}$ *will be the set* $[\,4\,]_3 = \{\cdots,-8,-5,-2,+1,+4,+7,\cdots\} = \{1+3.k|\forall k \in \mathbb{Z}\}$*. Select* $[\,1\,]_3$ *as identification of these integers. A similar argument for* $\zeta = 2+3.k$ *such as* $\zeta = 5$ *results in* $[\,5\,]_3 = \{\cdots,-4,-1,+2,+5,+8,\cdots\}$*. Choose* $[\,2\,]_3$ *for this set. At the end we have,* $\mathbb{Z}_3 = \{[\,0\,]_3, [\,1\,]_3, [\,2\,]_3\}$*.*

$\mathbb{Z}_3$ has **three** members.

- A general observation is that $card\,(\mathbb{Z}_n) = n$. That is $\mathbb{Z}_n$ has n members.

$z = kn + \alpha$

$\alpha = -n$ then $z = k.n - n = (k-1).n = k'.n + 0;$

$\alpha = -(n-1)$ then $z = k.n - (n-1) = (k-1).n + 1 = k'.n + 1;$

$\vdots$

$\alpha = -2$ then $z = k.n - 2 = (k-1).n + (n-2) = k'.n + (n-2);$

$\alpha = -1$ then $z = k.n - 1 = (k-1).n + (n-1) = k'.n + (n-1);$

$\alpha = 0$ then $z = k.n + 0;$

$\alpha = 1$ then $z = k.n + 1;$

$\alpha = 2$ then $z = k.n + 2;$

$\vdots$

$\alpha = n-1$ then $z = k.n + (n-1);$

$\alpha = n$ then $z = k.n + n = (k+1).n = k".n + 0;$

Hence, α has only values from 0 to $n-1$. Other values of α exhaust to the same values, keep repeating.

- Another observation is that $\bigcup \mathbb{Z}_n = \mathbb{Z}$
- Always $\mathbb{Z}_n \nsubseteq \mathbb{Z}$ and $\mathbb{Z}_n \not\supseteq \mathbb{Z}$.
- It is easily could be seen that there is a bijection between set $\mathbb{Z}_n$ and set $\{0, 1, 2, \ \ldots, n-1\}$.

Remark 2.6.3. $n\mathbb{Z}$: Here it is a good point to become familiar with the set $n\mathbb{Z}$ and contrast it carefully with $\mathbb{Z}_n$. It is a countably infinite subset of $\mathbb{Z}$ and is defined as, assuming $n \neq 0$,

$$n\mathbb{Z} = \{\ldots, -3n, -2n, -n, 0, +n, 2n, 3n, \ldots\}$$

That is, each integer is multiplied by an n, forming a subset of $\mathbb{Z}$. For example,

$$2\mathbb{Z} = \{\ldots, -6, -4, -2, 0, +2, 4, 6, \ldots\}$$

and,

$$5\mathbb{Z} = \{\ldots, -15, -10, -5, 0, +5, 10, 15, \ldots\}$$

and,

$$12\mathbb{Z} = \{\ldots, -36, -24, -12, 0, +12, 24, 36, \ldots\}$$

We always have,

$$n\mathbb{Z} \subseteq \mathbb{Z}$$

Please note that elements of $n\mathbb{Z}$ are not multiplications. In essence, they are results of **additions** of each member of $\mathbb{Z}$ added n times together.

Remark 2.6.4. Sometimes, you may encounter with a notation conveniently written as

$$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} \quad \text{(beautiful!)}$$

This means that you identify points of $\mathbb{Z}$ modulo n. In group theory this notation is helpful in recognizing the quotient groups.

In Example 2.6.5 we succeeded to build set of integers $\mathbb{Z}$ by defining an equivalence relation on $\mathbb{N}^\dagger \times \mathbb{N}^\dagger$. Here, we follow that approach and build the set of rational numbers $\mathbb{Q}$ from $\mathbb{Z} \times \mathbb{Z}^\dagger$. Again we have removed zero from $\mathbb{Z}$ and show the resulting set by $\mathbb{Z}^\dagger = \mathbb{Z} \setminus \{o\}$.

Example 2.6.10. *Set of Rational Numbers* $\mathbb{Q}$ *: Take* $(\mu,\nu)\in\mathbb{Z}^{\dagger}\times\mathbb{Z}^{\dagger}$ *and define relation shown with symbol* $\sim$ *by identifying all* $(m,n)\in\mathbb{Z}^{\dagger}\times\mathbb{Z}^{\dagger}$ *as* $(m,n)\sim(\mu,\nu)$ *if and only if* $m.\nu = n.\mu$*. Then the equivalence set (class)* $[(\mu,\nu)]$ *is said to be a rational number. The set of all rational numbers is shown by* $\mathbb{Q}$*. Note that* $[(\mu,\mu)]=1$ *and* $[(0,\nu)]=0$*.*

Example 2.6.11. *Order in the Set of Rationals* $\mathbb{Q}$ *: Remember we created a natural order in the set of integers by the* $\leq$ *relation. Take* $(\mu,\nu)\in\mathbb{Z}\times\mathbb{Z}$ *and define relation shown with symbol* $\leq$ *by all* $[(m,n)],[(\mu,\nu)]\in\mathbb{Q}$ *if and only if* $m\cdot\nu\leq n\cdot\mu$*. Then this relation is reflexive, antisymmetric, and transitive in ordering the equivalent classes of the form* $[(m,n)]$*; in the sense that, if* $q_1=[(m_1,n_1)]$ *and* $q_2=[(m_2,n_2)]$*. Then* $q_1\leq q_2$ *if and only if* $m_1\cdot n_2\leq m_2\cdot n_1$*.*

It is easy to verify that the 0, 1, and the order defined in rationals serves equivalently as the 0, 1, and the order in the integers.

2.7. Indexing Sets

Definiteion 2.7.1. *Net : Let* X *be a directed set (defined 2.3.10) then a mapping from* X *to the set* Ω *is called a net in* Ω*.*

Definiteion 2.7.2. *Indexing Map : Let* $\mathfrak{M}$ *be a collection of subsets of the set* Ω*. A surjective map* f *from a set* J*, called* ***index set****, to* $\mathfrak{M}$ *is said to be an indexing map.*

Definiteion 2.7.3. *Indexed Family of Sets : Let* $\mathfrak{M}$ *be a collection of subsets of the set* Ω*. Take the set* J *as an index set. Then the indexing surjection* $f:J\longrightarrow\mathfrak{M}$ *of* $\alpha\mapsto f(\alpha)$ *together with the family* $\mathfrak{M}$ *is called an indexed family of sets. We denote* $f(\alpha)$ *by* A_α*. Indexed family is shown by*$\{A_\alpha\}_{\alpha\in J}$

A J-tuple is very similar to an indexed family of sets. The difference is that it is defined on members of a set rather than subsets of a set.

Definiteion 2.7.4. *$J-$tuple : Assume X is a subset of the set Ω. Take the set J as an index set. Then a mapping $\mathbf{x}:J\longrightarrow X$ of $\alpha\mapsto\mathbf{x}(\alpha)$ is called a $J-$tuple. We denote $\mathbf{x}(\alpha)$ by x_α and we call it the $\alpha-$coordinate of $\mathbf{x}$. The collection shown by $\{x_\alpha\}_{\alpha\in J}$ is called coordinates of $\mathbf{x}$.*

Definiteion 2.7.5. *$J-$power : The set of all $J-$tuples of set X is called a $J-$power of X and is denoted by ${}^{J}X$ or X^{J}. $J-$power is a subset of the power set $\mathscr{P}(J\times X)$*

Definiteion 2.7.6. *Cartesian Product : Let* $\{A_\alpha\}_{\alpha\in J}$ *be an indexed family of subsets of* Ω*. Assume* $X=\bigcup_{\alpha\in J}A_\alpha$*. Then the Cartesian product* $\prod_{\alpha\in J}A_\alpha$ *is the set* $\{\mathbf{x}:J\to X\}$ *such that* $\mathbf{x}(\alpha)\in A_\alpha$*,* $\forall\alpha\in J$

Perhaps you remember from 1.4.2 already we promised to extend the idea of ordered pairs and the Cartesian products from X^n to any value of n. See, the Cartesian product is actually itself a set of mappings as defined in previous definition. Here we find an elegance of mathematical rigour for consistency. To further explain it, let me elaborate this definition, I have given here, to a set $J = \{1, 2\}$ and let $A_1 = \{\xi, \zeta\}$ and $A_2 = \{a, b, c\}$. Then we write

$$\begin{aligned}\{\mathbf{x}\} = \{ & \\ & \{(1, \xi), (2, a)\}, \{(1, \xi), (2, b)\}, \{(1, \xi), (2, c)\}, \\ & \{(1, \zeta), (2, a)\}, \{(1, \zeta), (2, b)\}, \{(1, \zeta), (2, c)\} \\ & \}.\end{aligned} \tag{2.7.1}$$

This is the set of all possible mappings conceivable on the set J. There are six possible mappings altogether, as I have separated them by putting each mapping in its own braces. Each mapping has two components in the braces, each component as an ordered set itself with first element from J and the second element from its related index: A_1 for $1 \in J$, or A_2 for $2 \in J$. The first thing we notice is having *six* members as already we expect for Cartesian product of $A_1 \times A_2 = \{(\xi, a), (\xi, b), (\xi, c), (\zeta, a), (\zeta, b), (\zeta, c)\}$. What is (ξ, a)? It says, ξ comes first and then comes a. Its equivalent is the mapping $\{(1, \xi), (2, a)\}$. First component of mapping is $(1, \xi)$, that is ξ comes first (tagged so with a 1), and second component of mapping is $(2, a)$, that is a comes second (tagged so by a 2). Another point that we notice is if we have a string, (ξa) and assume we have a pointer needle shown as a punctuation mark comma "," If we move the needle from left (beginning of strings as we write from left to right) to the right (end of the string), the comma first registers ξ as the set $\{\xi\}$ and then registers ξa as the set $\{\xi, a\}$. Hence we read the set $\{\{\xi\}, \{\xi, a\}\}$ altogether as its log. In shadow of this, we understand item (5) of 1.4.2 better if we consider $J = \{1\}$.

Further we can see what is a tuple, comparing it with the meaning that we know from Calculus course about coordinate system. We remember coordinates of a 1-dimensional space (e.g., a line) are sing**le**s, a 2-space are doub**le**s, a 3-space are trip**le**s, and for n-space they are n-tup**le**s. Here we notice that a J-tuple defined in 2.7.4 is consistent with that idea considering each one of the mappings we can separate in 2.7.1. Here, as a $J-$tuple we have a double coordinates for instance, $\{(1, \xi), (2, a)\}$. Its first coordinate is $\mathbf{x}(1) = \xi$, and its second coordinate is $\mathbf{x}(2) = a$, or $x_1 = \xi$ and $x_2 = a$.

DEFINITEION 2.7.7. *Box : Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of subsets of Ω. Further assume that there exists the Cartesian product $\prod_{\alpha \in J} A_\alpha$. Now assume $\forall \alpha \in J$ there exists $B_\alpha \subseteq A_\alpha$ Then the box B is defined as $\prod_{\alpha \in J} B_\alpha$*

A parameterizing map becomes important when we define a manifold.

DEFINITEION 2.7.8. *Parameterizing Map : Assume A is a parameter set. Consider the indexed family of sets $\{A_\alpha\}_{\alpha \in J}$ and an indexed family of mappings $\{f_\alpha\}_{\alpha \in J}$, both indexed with J such that $f_\alpha : A \longrightarrow A_\alpha$ then the mapping $f : A \longrightarrow \prod_{\alpha \in J} A_\alpha$ where $f(t) = (f_\alpha(t))_{\alpha \in J}$, $t \in A$ is called a parameterizing mapping on A.*

As an example, consider the time dependent system trajectory on an euclidean plane. We have index set $J = \{1, 2\}$ and parameter set of non-negative real numbers, $A = \{t | t \in \mathbb{R}^*\}$ such that, $t \mapsto (v_0 t, 0.5a.t^2)$. Here, $f_1(t) = v_0.t$ and $f_2(t) = 0.5a.t^2$, where v_0 is initial speed and a is the acceleration of the object.

DEFINITEION 2.7.9. *Projection Map : Take* $\{A_\alpha\}_{\alpha\in J}$ *as an indexed family of sets. The mapping* $\pi_\beta : \prod_{\alpha\in J} A_\alpha \longrightarrow A_\beta$ *where* $\pi_\beta\left((a_\alpha)_{\alpha\in J}\right) = a_\beta$ *is called a projection map on* A.

In view of 2.7.1 let's see what is, say, $\pi_2(\{(1, \xi), (2, a)\})$? It picks the second element of this mapping, the element that is accompanying $\beta = 2$. It is the element $a \in A_2$ from $(2, a)$. That is, $\pi_2(\{(1, \xi), (2, a)\}) = a$. You appreciate that projection mapping is not an injection at all. I write all possible values here,

$$\begin{aligned}
&\pi_1(\{(1, \xi), (2, a)\}) = \xi,\\
&\pi_1(\{(1, \xi), (2, b)\}) = \xi,\\
&\pi_1(\{(1, \xi), (2, c)\}) = \xi,\\
&\pi_1(\{(1, \zeta), (2, a)\}) = \zeta,\\
&\pi_1(\{(1, \zeta), (2, b)\}) = \zeta,\\
&\pi_1(\{(1, \zeta), (2, c)\}) = \zeta,\\
&\pi_2(\{(1, \xi), (2, a)\}) = a,\\
&\pi_2(\{(1, \xi), (2, b)\}) = b,\\
&\pi_2(\{(1, \xi), (2, c)\}) = c,\\
&\pi_2(\{(1, \zeta), (2, a)\}) = a,\\
&\pi_2(\{(1, \zeta), (2, b)\}) = b,\\
&\pi_2(\{(1, \zeta), (2, c)\}) = c,
\end{aligned} \tag{2.7.2}$$

Idea of bundles come from idea of product space.

DEFINITEION 2.7.10. *Product Space : Again let* $\{A_\alpha\}_{\alpha\in J}$ *be an indexed family of subsets of* Ω. *We can define* $S_\beta = \pi_\beta^{-1}\left((B_\beta)_{\beta\in J}\right)$ *for some* $B_\beta \subseteq A_\beta$ *and for all* $\beta \in J$. *Then* $S = \bigcup_{\beta\in J} S_\beta$ *is called the product space of the indexed family* $\{B_\alpha\}_{\alpha\in J}$.

Assume $\beta = 2.$ and $B_2 = \{a, c\} \subset A_2$. Then

$$\begin{aligned}
S_2 = \pi_2^{-1}(\{a, c\}) = \{ &\\
&\{(1, \xi), (2, a)\},\\
&\{(1, \xi), (2, c)\},\\
&\{(1, \zeta), (2, a)\},\\
&\{(1, \zeta), (2, c)\}\\
&\}
\end{aligned} \tag{2.7.3}$$

Further assume $\beta = 1.$ and $B_1 = \{\zeta\} \subset A_1$. Then

$$S_1 = \pi_1^{-1}(\{\zeta\}) = \{ \begin{array}{l} \{(1,\zeta),(2,a)\}, \\ \{(1,\zeta),(2,b)\}, \\ \{(1,\zeta),(2,c)\}, \\ \} \end{array} \tag{2.7.4}$$

Hence, we have,

$$S = S_1 \cup S_2 = \{ \begin{array}{l} \{(1,\xi),(2,a)\}, \\ \{(1,\xi),(2,c)\}, \\ \{(1,\zeta),(2,a)\}, \\ \{(1,\zeta),(2,b)\}, \\ \{(1,\zeta),(2,c)\} \\ \} \end{array} \tag{2.7.5}$$

Definiteion 2.7.11. *Disjoint Union : Take $\{A_\alpha\}_{\alpha\in J}$ as a collection of indexed family of sets. Then the set $\coprod A_\alpha = \bigcup_{\alpha\in J}\{(x,\alpha)\,|\forall x \in A_\alpha\}$ is called disjoint product of the collection of sets.*

This is how we bring elements of an indexed family of sets attached to and labeled with the index of their container sets into a union of all the sets of the family. It is like this for instance, for $\beta = 2$ and $A_1 = \{\xi, \zeta\}$ and $A_2 = \{a, b, c\}$. We have,

$$\coprod A_\alpha = \{ \begin{array}{l} (\xi,1),(\zeta,1), \\ (a,2),(b,2),(c,2) \\ \} \end{array} \tag{2.7.6}$$

This is the simple indexing we first encountered in the basic calculus. Question remains that why do we call it a disjoint union. Assume, now we have again $\beta = 2$ but $A_1 = \{\xi, \zeta, a\}$ and $A_2 = \{a, b, c\}$. Then the disjoint union of the two sets is,

$$\coprod A_\alpha = \{ \begin{array}{l} (\xi,1),(\zeta,1),(a,1), \\ (a,2),(b,2),(c,2) \\ \} \end{array} \tag{2.7.7}$$

We are assured that the two sets to be unioned are already disjoint by indexing their elements to their respective sets, in spite of having the common element a.

2.8. Cut

Up to this point we succeeded to build natural, integer, and rational numbers. Using a **cut** we are going to build *real* numbers.

DEFINITEION 2.8.1. *Dedekind Cut : A Dedekind cut is defined as a set* $\mathbf{x}$ *such that,*

(1) *It is a nonempty subset of* $\mathbb{Q}$*, that is,* $\varnothing \subsetneq \mathbf{x} \subsetneq \mathbb{Q}$.
(2) *The set* $\mathbf{x}$ *is closed downward, that is,* $\forall s \in \mathbf{x}$ *if* $r \in \mathbb{Q}$ *and* $r < s$ *then* $r \in \mathbf{x}$.
(3) $\forall s \in \mathbf{x}$, $\exists t \in \mathbf{x}$ *such that* $s < t$.

You remember that so far we defined the set of integers $\mathbb{Z}$ as a set of equivalence classes over the set of natural numbers $\mathbb{N}$ and the set of rationals $\mathbb{Q}$ as a set of equivalent classes over the set of integers $\mathbb{Z}$. That line does not continue to define the next stage, i.e., the set of real numbers $\mathbb{R}$. This set is defined as the set of all cuts, each cut $\mathbf{x}$ is defined as a subset of rational numbers. Each cut is one real number.
In other approaches, one might see other ways in analysis to build the set of real numbers. But building up from the empty set $\varnothing$ building stones to natural numbers and then to integers and then to rationals and at last defining a cut is the most straightforward mind-pleasing way of accomplishing this, and perhaps more axiomatics and rigorous approach.

Example 2.8.1. *Order in the Set of Real Numbers* $\mathbb{R}$ *: Remember we created a natural order in the set of rationals by the* $\leq$ *relation. Among the real numbers we say the real number* $\boldsymbol{x} < \boldsymbol{y}$ *if and only if* $\boldsymbol{x} \subset \boldsymbol{y}$.

CHAPTER 3

Measure Theory Structures

Assume Ω is a set and $\mathfrak{M}$ is a collection of subsets of Ω.

3.1. Semi-Ring Structures on Sets

DEFINITEION 3.1.1. *Von Neumann Semi-ring : A collection of subsets $\mathfrak{M}$ is a Von Neumann semi-ring if it satisfies two following conditions*

(1) *If $A, B \in \mathfrak{M}$ then $A - B \in \mathfrak{M}$.*
(2) *If $A, B \in \mathfrak{M}$ and $A \subseteq B$ then there is a chain $\{A_i | A_i \in \mathfrak{M}, i = 0, 1, \cdots, n\}$ such that $A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = B$ and $A_i - A_{i-1} \in \mathfrak{M}$ for $i = 1, \cdots, n$.*

DEFINITEION 3.1.2. *Semi-ring : A collection of subsets $\mathfrak{M}$ is a semi-ring if and only if*

(1) *If $A, B \in \mathfrak{M}$ then $A - B \in \mathfrak{M}$.*
(2) *If $A, B \in \mathfrak{M}$ then there is a countable partition $\{ A_i | A_i \in \mathfrak{M}, i = 0, 1, \cdots\}$ for $A - B$; that is $A - B = \bigcup_{i=1}^{\infty} A_i$.*

3.2. Ring Structures on Sets

DEFINITEION 3.2.1. *Ring (Boolean Ring or Finite Union Ring): A ring is defined as a collection of subsets where,*

(1) *If $A, B \in \mathfrak{M}$ then $A - B \in \mathfrak{M}$.*
(2) *If for any collection $\{ A_i | A_i \in \mathfrak{M}, i = 0, 1, \cdots, n\}$ we have $\bigcup_{i=1}^{n} A_i \in \mathfrak{M}$.*

DEFINITEION 3.2.2. *$\sigma-$ring (Infinite Union Ring): A $\sigma-$ring is defined as a collection of subsets where,*

(1) *If $A, B \in \mathfrak{M}$ then $A - B \in \mathfrak{M}$.*

(2) *If for any countable collection* $\{ A_i \,|A_i \in \mathfrak{M}, i = 0, 1, \cdots\}$ *we have* $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{M}$.

DEFINITEION 3.2.3. $\sigma - \mathfrak{M}$ *(Ring) : It is a collection* $\mathfrak{M}$ *of subsets of* Ω *where*

(1) $\mathfrak{M}$ *is a* $\sigma-$*ring.*
(2) $\exists \{ A_i \,|A_i \in \mathfrak{M}, i = 0, 1, \cdots\}$ *such that* $\Omega = \bigcup_{i=1}^{\infty} A_i$

3.3. Field Structures on Sets

DEFINITEION 3.3.1. *Field (Kuratowsky Field): A collection* $\mathfrak{M}$ *of subsets of* Ω *is a field whenever,*

(1) $\mathfrak{M}$ *is a ring and*
(2) $\Omega \in \mathfrak{M}$

DEFINITEION 3.3.2. $\sigma-$*field (Borel Field): A collection* $\mathfrak{M}$ *of subsets of* Ω *is a* $\sigma-$*field whenever,*

(1) $\mathfrak{M}$ *is a* $\sigma-$*ring and*
(2) $\Omega \in \mathfrak{M}$

DEFINITEION 3.3.3. $\sigma - \mathfrak{M}(Field)$ *: It is a collection* $\mathfrak{M}$ *of subsets of* Ω *where*

(1) $\mathfrak{M}$ *is a* $\sigma-$*field.*
(2) $\exists \{ A_i \,|A_i \in \mathfrak{M}, i = 0, 1, \cdots\}$ *such that* $\Omega = \bigcup_{i=1}^{\infty} A_i$

3.4. Algebra Structures on Sets

DEFINITEION 3.4.1. *Algebra (Boolean Algebra): An algebra is defined as a collection of subsets where,*

(1) *If* $A \in \mathfrak{M}$ *then* $A^c \in \mathfrak{M}$.
(2) *If for any finite collection of subsets* $\{ A_i \,|A_i \subseteq \Omega, i = 0, 1, \cdots, n\}$ *of* Ω *where* $\bigcup_{i=1}^{n} A_i \in \mathfrak{M}$ *then we have each* $A_i \in \mathfrak{M}$.
(3) $\Omega \in \mathfrak{M}$

DEFINITEION 3.4.2. $\sigma-$*algebra : A* $\sigma-$*algebra is defined as a collection of subsets where,*

(1) *If* $A \in \mathfrak{M}$ *then* $A^c \in \mathfrak{M}$.
(2) *If for any countable collection of subsets* $\{ A_i \,|A_i \subseteq \Omega, i = 0, 1, \cdots\}$ *of* Ω *where* $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{M}$ *then we have each* $A_i \in \mathfrak{M}$.
(3) $\Omega \in \mathfrak{M}$

DEFINITEION 3.4.3. $\sigma - \mathfrak{M}(Algebra)$ *: It is a collection* $\mathfrak{M}$ *of subsets of* Ω *where*

(1) $\mathfrak{M}$ *is a* $\sigma-$*algebra.*
(2) $\exists \{ A_i \,|A_i \in \mathfrak{M}, i = 0, 1, \cdots\}$ *such that* $\Omega = \bigcup_{i=1}^{\infty} A_i$

3.5. Sequences of Sets

DEFINITEION 3.5.1. *Increasing Sequence of Sets: For* $A_i \in \mathfrak{M}$, *and* $\forall i \in \mathbb{N}$ *we have,* $A_i \subseteq A_{i+1}$.

DEFINITEION 3.5.2. *Decreasing Sequence of Sets: For* $A_i \in \mathfrak{M}$, *and* $\forall i \in \mathbb{N}$ *we have,* $A_i \supseteq A_{i+1}$.

DEFINITEION 3.5.3. *Monotone Sequence of Sets: A sequence of Sets is Monotone when it is either an increasing or a decreasing sequence of sets.*

DEFINITEION 3.5.4. *Monotone Class : A monotone class is defined as a countable collection* $\mathfrak{M}$ *of subsets* $\{ A_i \,|A_i \subseteq \Omega, i = 0, 1, \cdots\}$ *where,*

(1) *If* $A_i \subseteq A_{i+1}$ *(or alternatively* $A_i \supseteq A_{i+1}$*) for each* $i = 0, 1, \cdots$, *and*
(2) $\bigcup_{i=0}^{\infty} A_i \in \mathfrak{M}$ *(or alternatively* $\bigcap_{i=0}^{\infty} A_i \in \mathfrak{M}$*).*

Already we defined supremum 2.3.4 and infimum of any collection of sets. Here we limit its scope to a countable collection of sets.

DEFINITEION 3.5.5. *Supremum of a Sequence of Sets: Assume I have any sequence of sets*

$$\{A_i \,|\, A_i \in \mathfrak{M} \, for \, i = 0, 1, \cdots\}.$$

Then the supremum of a subsequence $\{A_i \in \mathfrak{M} \, for \, i = p, p+1, \cdots\}$ *for any* $p \in \mathbb{N}$ *is defined as the set sequence* $\left\{\mathbf{sup}_p\right\} = \left\{\bigcup_{i=p}^{\infty} A_i\right\}$.

DEFINITEION 3.5.6. *Infimum of a Sequence of Sets: Assume I have any sequence of sets*

$$\{A_i \,|\, A_i \in \mathfrak{M} \, for \, i = 0, 1, \cdots\}.$$

Then the infimum of a subsequence

$$\{A_i \in \mathfrak{M} \, for \, i = p, p+1, \cdots\}$$

for any $p \in \mathbb{N}$ *is defined as the set sequence* $\{\mathbf{inf}_p\} = \left\{\bigcap_{i=p}^{\infty} A_i\right\}$.

DEFINITEION 3.5.7. *Limit Supremum of a Sequence of Sets: Assume I have any sequence of sets* $\{A_i \,|\, A_i \in \mathfrak{M} \; for \; i = 0, 1, \cdots\}$. *Then the limit supremum of that sequence* $\{A_i \in \mathfrak{M} \; for \; i = 0, \, 1, \, \cdots\}$ *is defined as*

$$\lim_{n\to\infty} \; \mathbf{sup}\,\{A_i\} = \bigcap_{p=0}^{\infty} \left\{\mathbf{sup}_p\right\}; \; \forall p \in \mathbb{N}.$$

DEFINITEION 3.5.8. *Limit Infimum of a Sequence of Sets: Assume I have any sequence of sets* $\{A_i \,|\, A_i \in \mathfrak{M} \; for \; i = 0, 1, \cdots\}$. *Then the limit infimum of that sequence* $\{A_i \in \mathfrak{M} \; for \; i = 0, \, 1, \, \cdots\}$ *is defined as*

$$\lim_{n\to\infty} \; inf\,\{A_i\} = \bigcup_{p=0}^{\infty} \{\mathbf{inf}_p\}; \; \forall p \in \mathbb{N}.$$

DEFINITEION 3.5.9. *Limit of a Sequence of Sets : If in a sequence of sets, the limit supremum is equal to the limit infimum then that sequence is said to have a limit,* $\lim_{n\to\infty}\{A_i\} = \lim_{n\to\infty} \; \mathbf{sup}\,\{A_i\} = \lim_{n\to\infty} \; \mathbf{inf}\,\{A_i\}$. *A sequence that has limit said to converges or being convergent. If a sequence is not convergent then it is called divergent.*

Example 3.5.1. *Every monotone sequence of sets has limit : Assume* $\{A_i\}$ *is an increasing sequence of sets. Then*

$$\left\{\mathbf{sup}_p\right\} = \left\{\bigcup_{i=p}^{\infty} A_i\right\} \stackrel{?}{=} \left\{\bigcup_{i=0}^{\infty} A_i\right\} \quad and \quad \left\{\mathbf{inf}_p\right\} = \left\{\bigcap_{i=p}^{\infty} A_i\right\} \stackrel{?}{=} \{A_p\} \quad for \; all \, p.$$

As $\bigcup_{i=0}^{\infty} A_i$ *is independent of* p *then* $\lim_{n\to\infty} \; \mathbf{sup}\,\{A_i\} = \bigcap_{p=0}^{\infty}\{\bigcup_{i=0}^{\infty} A_i\} = \bigcup_{i=0}^{\infty} A_i$. *On the other hand,* $\lim_{n\to\infty} \; \mathbf{inf}\,\{A_i\} = \bigcup_{p=0}^{\infty}\{A_p\} = \bigcup_{i=0}^{\infty} A_i$. *A similar proof can be applied to a decreasing sequence of sets. Please, deliberate on the points shown by "*$\stackrel{?}{=}$*".*

Example 3.5.2. *Not every non-monotone sequence of sets is divergent: a counter -example is*

$$\{A_i\} = \begin{cases} \{x \,|\, 0 < x \leq 1 - (1/i)\} & if \; i \;\; odd \\ \{x \,|\, (1/i) \leq x < 1\} & if \; i \;\; even, \end{cases}$$

First show that this is neither an increasing nor a decreasing sequence of sets. Then prove that it is convergent

AXIOM 3. *Axiom of Choice : Given a non-empty class* $\mathfrak{M}$ *of disjoint sets* A_i, *then there exists a set* $B \subset \bigcup\{A_i \,|\, A_i \in \mathfrak{M}\}$ *such that* $B \cap A_i$ *is a single point set for each* $A_i \in \mathfrak{M}$.

In other words, we can take (choose) arbitrarily one point from each set and make an arbitrary new set B out of those points.

AXIOM 4. *Axiom of Choice : For a non-empty class* $\mathfrak{M}$ *of disjoint sets* A_i, *there exists a mapping (called a choice mapping)* $f : \mathfrak{M} \to \bigcup\{A_i \,|\, A_i \in \mathfrak{M}\}$ *such that, for each* $A_i \in \mathfrak{M}$, *we have,* $f(A_i) \in A_i$

In other words, there is a mapping that can take (choose) an arbitrary point from each set.

CHAPTER 4

Algebraic Structures

4.1. Rudiments

DEFINITEION 4.1.1. *Binary Operation on a Set : Assume S is a set. A binary operation on two elements $a \in S$ and $b \in S$ is a mapping on $S \times S$. We show a binary operation by $\mathfrak{O}$ or $\circ$ or $\bullet$ or $\cdot$ or $*$. The result of operation is shown as $\mathfrak{O}(a, b)$ or $a \circ b$ or $a \bullet b$ or $a \cdot b$ or $a * b$, respectively. We choose $a \circ b$ as our standard notation for our binary operations. When there is no ambiguity we even prefer to use ab as the result of binary operation $\mathfrak{O}(a, b)$.*

DEFINITEION 4.1.2. *n-ary Operation on a Set : n-ary operation on a set S can be defined recursively as a binary operation on $S \times S^{n-1}$.*

DEFINITEION 4.1.3. *Structure : When we have a binary operation $\mathfrak{O}$ on set S, we say that the binary operation defines, or actually recognizes and reveals a structure $[S, \mathfrak{O}]$ on set S.*

DEFINITEION 4.1.4. *Closed : If the result of a binary operation on a set S is a member of the set S then we say the set S is **closed** under that specific binary operation.*

DEFINITEION 4.1.5. *Commutative : When $a \circ b$ and $b \circ a$ result in the same, we say that the binary operation is **commutative** and write it as $a \circ b = b \circ a$.*

DEFINITEION 4.1.6. *Associative : When $(a \circ b) \circ c$ and $a \circ (b \circ c)$ result in the same, we say that the binary operation is **associative** and write it as $(a \circ b) \circ c = a \circ (b \circ c) = a \circ b \circ c$.*

DEFINITEION 4.1.7. *Right Cancellation Law: We say the binary operation $\circ$ on set S follows the right cancellation rule if from the $a \circ c = b \circ c$ we can get to the $a = b$ for all the $a, b, c \in S$.*

DEFINITEION 4.1.8. *Left Cancellation Law: We say the binary operation $\circ$ on set S follows the left cancellation rule if from the $c \circ a = c \circ b$ we can get to the $a = b$ for all the $a, b, c \in S$.*

DEFINITEION 4.1.9. *Regular Element: When in a set S for a binary operation $\circ$ we can find an element s that satisfies both cancellation laws we call that element a regular element.*

DEFINITEION 4.1.10. *Unity (neutral element): When in a set S for a binary operation $\circ$ we can find an element e such that $\forall s \in S$ we can have $e \circ s = s \circ e$ then we call e the **unity** or **neutral** element of S for binary operation $\circ$.*

DEFINITEION 4.1.11. *Unit (inverse element) : Assume $\circ$ is a binary operation defined in set S. Further assume that S is endowed with the unity element e. If for an $s \in S$ we can find an element u such that $u \circ s = s \circ u = e$. Then s is called a **unit** element of S. s and u are called **inverse** of each other.*

DEFINITEION 4.1.12. *Involution (involutary element): When in a set S for a binary operation $\circ$ we can find an element s such that we can have $s \circ s = e$ then we call s the **involution** or **involutary** element of S for binary operation $\circ$.*

DEFINITEION 4.1.13. *Nilpotent Element: For any $s \in S$ we can define s^n recursively as $s^n = s \circ s^{n-1}$. If we can find an element $s \in S$ such that we can have $s^n = e$ then we call s the **nilpotent** element of S for binary operation $\circ$.*

DEFINITEION 4.1.14. *Idempotent Element: When in a set S for a binary operation $\circ$ we can find an element s such that we can have $s \circ s = s$ then we call s the **idempotent** element of S for binary operation $\circ$.*

Next we assume that f is a mapping from set X to set Y. Also we have binary operation $\circ$ on X and binary operation $\bullet$ on Y

DEFINITEION 4.1.15. *Morphism (Structure Preserving Mapping) :A morphism f from set X to set Y carries structure of X into the structure of Y. That is, $f(a \circ b) = f(a) \bullet f(b)$*

(1) *HomoMorphism: This is a morphism from a set X into a different set Y.*
(2) *EndoMorphism: EndoMorphism is a morphism from a set X into the same set X.*

DEFINITEION 4.1.16. *Kernel of Morphism : Assume f is a morphism from set X to set Y. Then the set $\forall x \in X$ such that $f(x) = 0$ is called the kernel of morphism f.*

DEFINITEION 4.1.17. *Homomorphism : When the mapping $f : X \longrightarrow Y$ is just an **into** mapping.*

DEFINITEION 4.1.18. *Monomorphism : When that mapping is an injection mapping then the homomorphism is a monomorphism*

DEFINITEION 4.1.19. *Epimorphism : When the mapping is a surjection mapping then the homomorphism is an epimorphism.*

DEFINITEION 4.1.20. *Isomorphism : If the mapping is both an injection and a surjection then the homomorphism is an isomorphism.*

DEFINITEION 4.1.21. *Endomorphism : When the mapping $f : X \longrightarrow X$ is just an **into** mapping.*

DEFINITEION 4.1.22. *Endo-monomorphism: When that mapping is an injection mapping then the endomorphism is an endo-monomorphism*

DEFINITEION 4.1.23. *Endo-epimorphism : When the mapping is a surjection mapping then the endomorphism is an endo-epimorphism.*

DEFINITEION 4.1.24. *Automorphism (Endo-isomorphism): If the mapping is both an injection and a surjection then the endomorphism is an automorphism.*

4.2. Groups

DEFINITEION 4.2.1. *Groupoid : A set G with a binary operation defined on elements of that set is called a groupoid. A groupoid is a binary collection of the set G and the operator $\mathfrak{O}$; that is, $(G, \mathfrak{O})$.*

DEFINITEION 4.2.2. *Semi-group : Semi-group is a groupoid $(G, \mathfrak{O})$ where $\mathfrak{O}$ is **associative**.*

DEFINITEION 4.2.3. *Monoid : Monoid is a semi-group $(G, \mathfrak{O})$ where a **unity** element $e \in G$ exists for the binary operation $\mathfrak{O}$.*

DEFINITEION 4.2.4. *Group : Group is a monoid $(G, \mathfrak{O})$ where each element $g \in G$ is a* **unit** *with respect to the binary operation $\mathfrak{O}$.*

DEFINITEION 4.2.5. *Abelian Group (Commutative Group) : In group $(G, \mathfrak{O})$ the binary operation $\mathfrak{O}$ could be a commutative operator. In that case the group $(G, \mathfrak{O})$ is called a **commutative** or **Abelian** group*

DEFINITEION 4.2.6. *Subgroup : In group $(G, \mathfrak{O})$ let $H \subset G$. Assume $a, b \in H$. If $a \circ b^{-1} \in H$, as well, then H is a **subgroup** of G.*

DEFINITEION 4.2.7. *Subgroup Generated by a Subset : In group $(G, \mathfrak{O})$ let $X \subset G$. X is not necessarily a subgroup. Assume a set Y consists of all elements p formed from n elements $x_1, x_2, \cdots, x_n$, not necessarily different, taken from X such that $p = x_1^{\epsilon_1} \circ x_2^{\epsilon_2} \circ \cdots \circ x_n^{\epsilon_n}$ where $\epsilon_j = \pm 1$, $\forall j = 1, \cdots, n$. This new set Y has structure of a group and is called subgroup of G generated by X. We show this set by $gp(X)$.*

DEFINITEION 4.2.8. *Finitely Generated Subgroup : When X is a finite set with n elements the subgroup generated by X is called a **finitely** generated subgroup. This set, usually is shown by C_n, and is said to be **cyclic** group of order n.*

DEFINITEION 4.2.9. *Cyclic Subgroup : Assume the set X is a singleton $\{x\}$, then the finitely generated subgroup is called the **cyclic** group generated by x. We show it by $gp(\{x\})$.*

Definiteion 4.2.10. *Cosets (Left): Let H be a subgroup of G. Take $g \in G$. Then the set of all elements formed as gh, $\forall h \in H$ is called the **left coset** of the subgroup H and is denoted as gH.*

Definiteion 4.2.11. *Cosets (Right): Let H be a subgroup of G. Take $g \in G$. Then the set of all elements formed as hg, $\forall h \in H$ is called the **right coset** of the subgroup H. Right coset of H is shown as Hg.*

Definiteion 4.2.12. *Index of a Subgroup : Number of right cosets of H in G is said to be the index of H in G and is shown as $[G : H]$.*

Definiteion 4.2.13. *Quotient **of** Groups : Let G be a group and H a subgroup of G. A partition of G by left cosets of H is called **quotient** of group G by the subgroup H. We show the resultant partition by G/H.*

It can be proved that G/H is a partition for G.

Example 4.2.1. *Quotient $\mathbb{R}/\mathbb{Z}$: We want to find quotient of additive group $\mathbb{R}$ to its subgroup $\mathbb{Z}$. It means finding all cosets $x\mathbb{Z}$ for $x \in \mathbb{R}$. To appreciate further, I use the additive notation for showing cosets in form of $x + \mathbb{Z}$. In other words, put all $z \in \mathbb{Z}$ in the first coset which is actually $\mathbb{Z}$ subgroup itself; then take the next $x \in \mathbb{R}$ in $0 < x \leq 1$ and put all $x + z$, $\forall z \in \mathbb{Z}$ in the next coset, and continue until you get to $x = 1$. You will find out that for $x = 1$ you get to the first coset $\mathbb{Z}$. All the cosets after $x > 1$ will also become repeated, coinciding with the previous corresponding cosets, as if round and round around a circle. Compare this with the quotient in 2.6*

Definiteion 4.2.14. *Normal Subgroup : Let H be a subgroup of G. H is called a normal subgroup when $g^{-1}hg \in H$, $\forall g \in G$ and $\forall h \in H$. To show H as a normal subgroup of G we use notation $H \lhd G$.*

Definiteion 4.2.15. *Simple Group : A group without a **proper normal** subgroup is a simple group.*

Definiteion 4.2.16. *Factor Group : Let G be a group and H a **normal** subgroup of G. One can prove that the resulting quotient G/H is a **group** and a subgroup of G. G/H is called the **factor** group.*

Definiteion 4.2.17. *Commutator : Let x, $y \in G$, where G is a group. Then **commutator** w of x and y is defined as $w = x^{-1}y^{-1}xy$ we show the **commutator** by the bracket notation $w = [x, y]$.*

Definiteion 4.2.18. *Commutator Subgroup : The subgroup G' of G generated by $[x, y]$, that is, $gp([x, y])$ is called the commutator subgroup of G.*

We show that G' is normal in G.

Definiteion 4.2.19. *Center of a Group : This is the set of those elements $a \in G$ such that $\forall g \in G$ we have $ag = ga$. We show the **center** of G by $A(G)$.*

DEFINITEION 4.2.20. *Centralizer : Let $A \subset G$, where A is not necessarily a subgroup of G. This time select those elements $g \in G$ such that $\forall a \in A$ we have $ag = ga$. This new set built by help of A is called the* ***centralizer*** *of A in G and is shown by $C(A)$.*

DEFINITEION 4.2.21. *Normalizer : Let $A \subset G$, where A is not necessarily a subgroup of G. Again select those elements $g \in G$ such that we have $Ag = gA$. This set built by help of A is called the* ***normalizer*** *of A in G and is shown by $N(A)$.*

DEFINITEION 4.2.22. *Normalizer of a subgroup H: Let $A \subset G$, where A is not necessarily a subgroup of G, and H be a subgroup of G. Select those elements $h \in H$ such that we have $h = h^{-1}Ah$. This set built by help of A and H is called the* ***normalizer*** *of A in subgroup H and is shown by $N_H(A)$.*

DEFINITEION 4.2.23. *Subnormal Series : In the chain $\{e\} = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = G$ of group G we have $A_i \lhd A_{i+1}$*

DEFINITEION 4.2.24. *Factors of a Subnormal Series : Quotients A_{i+1}/A_i are said to be the factors of a subnormal series.*

DEFINITEION 4.2.25. *Upper Central Series : A chain of subgroups $\{e\} = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = G$ of group G is an* ***upper central series*** *whenever A_1 is the center of G and A_{i+1}/A_i is the center of G/A_i*

In this case $A_{i+1} \lhd G$

DEFINITEION 4.2.26. *Nilpotent Group : Upper central series $\{e\} = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = G$ of group G is a finite chain.*

DEFINITEION 4.2.27. *Solvable Group : In the subnormal series the index of H in G that is, $[A_{i+1} : A_i]$ is a prime depending on i .*

DEFINITEION 4.2.28. *Composition Series : In the subnormal series A_{i+1}/A_i is a* ***simple*** *group; that is, it is a group without any* ***proper*** *normal subgroup.*

DEFINITEION 4.2.29. *H-conjugates of Subsets of a Group : Assume T and S are any two subsets of G and H is a subgroup of G; such that there exists $h \in H$ that implies $h^{-1}Sh = T$. Then S and T are called H-conjugates subsets of group G.*

DEFINITEION 4.2.30. *Conjugate Subsets : Let T and S to be any two subsets of G. Further, assume that there exists $g \in G$ such that $g^{-1}Sg = T$. Then S and T are called conjugates subsets of group G.*

DEFINITEION 4.2.31. *p-group : When the order of a group G is a power of a prime number p; that is, $|G| = p^r$ where r is a positive integer.*

DEFINITEION 4.2.32. *Sylow p-subgroup :*

DEFINITEION 4.2.33. *Ordered Group : Assume G is an Abelian group endowed with an order structure $\leq$. Then it is said to be an ordered group if for all $z \in G$, $a \leq b$ implies $a + z \leq b + z$. In an ordered group any element $e \leq a$ is called a positive element.*

DEFINITEION 4.2.34. *Riesz Group : Assume ordered group G is the subset of the ordered set X. G is said to be a Riesz group if for all $a, b \in G$ we have* $\sup(a, b) \in G$ *and* $\inf(a, b) \in G$.

Note that that in the Riesz group we only consider the supremum and infimum of a pair of elements. Supremum is the least upper bound. **An** upper bound of a subset A of elements of a set X could be any element in X that is larger than all elements of A. Hence, generally upper bounds constitute a subset of elements of X. Assume we find an $x \in B$ such that x is less than any element of B (it is easy to verify that such an x exists), then this x is the least upper bound of the set A; yes, the set A. This could belong to A or not belong to A. One should not confuse it with the "maximum" element of a subset. Maximum element could exist or not to exist. If exists it should belong to that subset. If the suprimum belongs to a set then it is also the maximum element of that set. At this point please have a look at the definition of "cut" (2.8.1).

4.3. Action of a Group

DEFINITEION 4.3.1. *Action of a Group on a Set: Assume there is a mapping, shown with symbol $\odot$, such that $\odot : G \times X \longrightarrow X$. Mapping $\odot$ is called the action of G on set X.*

DEFINITEION 4.3.2. *G-set of a Set : Let $\odot$ be an action of group G on set X. We say X is G-set if $\forall x \in X$ we have*

(1) *$e \odot x = x$ where e is unity in G.*
(2) *$(g_1 * g_2) \odot x = g_1 \odot (g_2 \odot x)$; $\forall g_1$ and $\forall g_2 \in G$, where $*$ is the binary operation in G.*

*Elements of G are also called **operators** on X when the context requires it.*

Note that in both definitions above, we took two members from G and X respectively in that order and G is on the **left** of X. The result is an element in X. We equally could define an action of G on the **right** of X.

DEFINITEION 4.3.3. *X_g−set : Let $\odot$ be an action of group G on set X. A subset Γ of X is said to be X_g−set if $\forall x \in \Gamma$ we have $g \odot x = x$ where g is an element in G.*

DEFINITEION 4.3.4. *G_X-group : Let $\odot$ be an action of group G on set X. A subset H of G is said to be G_x-set or **isotropic** subgroup of G if $\forall g \in H$ we have $g \odot x = x$ where x is an element in X.*

4.4. Rings

DEFINITEION 4.4.1. *Array of Binary Operations : An array of binary operations is a set of two or more binary operations defined to create an algebraic structures. We show it with a bracket, e.g., like this $[*, \odot]$.*

DEFINITEION 4.4.2. *Ringoid : This is defined as a structure built out of a set R and a duet array $[+, \cdot\,]$ of binary operations such that*

1. *$(R, +)$ is an Abelian (commutative) group*
2. *$(R, \cdot\,)$ is a groupoid.*
3. *$\cdot$ is distributive over $+$, both from the left and from the right. That is $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$.*

We show the ringoid as $(R, [+, \cdot\,])$

We use **'addition'** and **'multiplication'**, respectively to name the binary operations $[+, \cdot\,]$.

DEFINITEION 4.4.3. *Ring : A ring is a ringoid $(R, [+, \cdot\,])$ where $(R, \cdot\,)$ is a semi-group.*

DEFINITEION 4.4.4. *Unitary Ring : A unitary ring is a ring $(R, [+, \cdot\,])$ where $(R, \cdot\,)$ is a monoid.*

In any of the above three structures we can define new structures such that the multiplication to become commutative. A more interesting case is when you impose a constraint for multiplication to obey right (left) cancellation law as well.

DEFINITEION 4.4.5. *Integral Domain : An integral domain is a ring (ringoid, ring, or unitary ring) where $a \cdot b = 0$ implies either $a = 0$ or $b = 0$ or both $a = b = 0$.*

An integral domain usually is called a **domain**. Hence, as it is said, in an integral domain we have **no zero divisor**. If a and b are two nonzero elements of domain R then their product ab is nonzero.

DEFINITEION 4.4.6. *Division Ring : A division ring is a unitary ring $(R, [+, \cdot\,])$ where $(R, \cdot)$ is a group.*

DEFINITEION 4.4.7. *Skew Field : This is an old fashioned name for the division ring.*

Note that with the definition of division ring it is implied that the division ring is a ring with **unit** elements for multiplication.

DEFINITEION 4.4.8. *Field : A field is a division ring* $(R, [+, \cdot])$ *where* $(R, \cdot)$ *is an abelian (commutative) group.*

You note that a field is two abelian groups $(R, +)$ and $(R, \cdot)$ interwoven together through the distributive law of the underlying ringoid.

4.5. Ideals

Before proceeding to ideals, it is instructive to become familiar with a subring. That makes us able to clearly contrast it with an ideal. Concept of ideals has roots in determinants, multiplication of polynomials and symmetric polynomials.

DEFINITEION 4.5.1. *Subring : A subring* S *of the ring* R *is a subset of* R *such that*

(1) $(S, +)$ *is a subgroup of Abelian group* $(R, +)$.
(2) *If* $a \in S$ *and* $b \in S$ *then* $a \cdot b \in S$.
(3) $1 \in S$.

- The first item above is equivalent to proposition that $\forall a,\ b \in S$ we have $a - b \in S$, which then implies $0 \in S$.

Ideals bring the concept of group cosets to multiplication in rings. During this discussion we do not assume that the ring R is a commutative ring. We also avoid using right or left. We believe context is clear. All notations are written "right" sided.

DEFINITEION 4.5.2. *Ideal : Let* $(R, [+, \cdot])$ *is a ring. An ideal* I *is a subset of* R *such that*

(1) $(I, +)$ *is a subgroup of Abelian group* $(R, +)$ *and*
(2) *If* $i \in I$ *and* $r \in R$ *then* $i \cdot r \in I$.
(3) *Usually* $1 \notin I$.

- The first item (1) above is equivalent to proposition that $\forall a,\ b \in I$ we have $a - b \in I$, which then implies $0 \in I$.

- The second item (2) above is equivalent to proposition that $I.r \subseteq I, \forall r \in R$.
- R is an ideal in R.
- R is the **only** ideal in R that is a subring. No other ideal is a subring of R.
- $\{0\}$ is an ideal in R.
- The intersection of any family of ideals in R is an ideal in R.
- The third item (3) above is mentioned to help the reader to compare an ideal with a subring. There is only one ideal that defies this restriction as we see in the next definition.
- You might have noticed that a ring as we have define is not commutative in its multiplicative $(R, \cdot)$ semi-group. Hence we can have different left and right ideals. That conciseness is most of the time beyond the rigor considered for our exposure of the subject. If necessary we will mention it explicitly. We have tried to be consistent all the time.

DEFINITEION 4.5.3. *Proper Ideal : An ideal I in a ring R is a proper ideal if $I \neq R$.*

It is routine to show that in a proper ideal I, we always have $1 \notin I$
An ideal which is $\{0\}$ or R is said to be a **trivial** ideal. Otherwise, it is a **non-**trivial ideal.

DEFINITEION 4.5.4. *Simple Ring: A ring R which has **no** ideal but trivial ideals $\{0\}$ and R is called a simple ring.*

Hence a simple **ring** has only trivial **ideals**.

4.6. Arithmetic of Ideals

Ideals are a crucial point in understanding of a large part of algebra. We continue to make ourselves more friendly with concepts surrounding them. Though a bit artificial, arithmetic of ideals is important pedagogically and perhaps not much of later usage. As sets ideals have unions and intersection and other set theory operation that readers can work them out.
Addition is the set of term-by-term additions

DEFINITEION 4.6.1. *Addition of Ideals : Assume $\mathfrak{a}$ and $\mathfrak{b}$ are ideals. Then we define addition of $\mathfrak{a}$ and $\mathfrak{b}$ as,*

$$\mathfrak{a} + \mathfrak{b} = \{a + b \mid a \in \mathfrak{a} \text{ and } b \in \mathfrak{b}\}$$

DEFINITEION 4.6.2. *Multiplication of Ideals : Assume $\mathfrak{a}$ and $\mathfrak{b}$ are ideals. Then we define multiplication of $\mathfrak{a}$ and $\mathfrak{b}$ as,*

$$\mathfrak{a}\mathfrak{b} = \left\{ \sum_{i=1}^{n} a_i b_i \mid a_i \in \mathfrak{a} \text{ and } b_i \in \mathfrak{b},\ n \in \mathbb{Z}^+ \right\}$$
$$= \{a_1 b_1,\ a_1 b_1 + a_2 b_2,\ a_1 b_1 + a_2 b_2 + a_3 b_3,\ \dots\}$$

DEFINITEION 4.6.3. *Ideal Generated by a Subset of a Ring : Assume X is any subset of ring R. Consider the family of all ideals $I_\alpha, \alpha \in J$ in R such that $X \subset I_\alpha, \forall \alpha \in J$. Then $\bigcap_{\alpha\in J} I_\alpha$ is an ideal in R and we have $X \subset \bigcap_{\alpha\in J} I_\alpha$. We call $\bigcap_{\alpha\in J} I_\alpha$ the ideal generated by X and show it as (X).*

DEFINITEION 4.6.4. *Principal Ideal (Ideal generated by an element of a ring) : If X is a singleton set $\{\rho\} \subset R$ then (ρ) is called the **principal ideal** generated by ρ.*

One can show that $(\rho) = \{x | x = \rho r; \forall r \in R\} = \rho R$.

Remark 4.6.1. You immediately appreciate that $\rho = 1$ cannot generate a proper ideal.

Remark 4.6.2. Scaling of Ring : (ρ) is the **scale** up of the ring R. That is $(\rho) = \rho R$. You can bring the idea of **translate** from cosets of a subgroup to here to create **quotient rings**.

Similarly if $\{a_1, \cdots, a_n\}$ is a subset of R then

$$(a_1, \cdots, a_n) = \{a_1 r_1 + \cdots + a_n r_n \,;\ \forall r_i \in R; i = 1, \cdots, n\}$$

is the ideal generated by n–element set $\{a_1, \cdots, a_n\}$. In forming those sums we are taking n arbitrary elements from the ring R in each summation.

DEFINITEION 4.6.5. *Cosets of Ideal I with respect to group $(R, +)$: These cosets are defined in usual group notion as $I + r$, where $r \in R$.*

Note that $(R, +)$ is commutative. So there is no difference between right and left cosets.

DEFINITEION 4.6.6. *Quotient Group R/I : This quotient is defined in a natural way for construction of commutative group $(R/I, \oplus)$ by*

$$(I + r) \oplus (I + r') = I + (r + r') \tag{4.6.1}$$

Zero element of this quotient group is $I + 0 = I$

4.7. Quotient Rings

DEFINITEION 4.7.1. *Group Natural Map : Is defined as $\pi : (R, +) \longrightarrow (R/I, \oplus)$ such that $r \mapsto I + r$.*

In this sense π is a surjective group homomorphism. Now we are ready to grasp the concept of quotient ring by defining a multiplication in group $(R/I, \oplus)$ to construct a minimum structure of a semi-group

Definiteion 4.7.2. *Semi-group* $(R/I, \otimes)$ *: Define multiplication* $\otimes$ *as*

$$(I + r) \otimes (I + r') = I + (rr') \tag{4.7.1}$$

It is straightforward to check that the resulting structure satisfies a semi-group structure. For the next definition to be correct, we have to assume that I is a **proper** ideal of R.

Definiteion 4.7.3. *Quotient Ring* $(R/I, [\oplus, \otimes])$ *: Quotient structure constructed by hinging commutative group* $(R/I, \oplus)$ *and semi-group* $(R/I, \otimes)$ *shows structure of a ring. It is called quotient ring or* ***residue*** *of* R ***modulo*** *ideal* I*.*

Members of the quotient rings constructed in this way are in the form of $I + r$. Note that the natural group quotient map now can be extended as the natural ring map from ring R to quotient ring R/I. and we have,

$$\pi(r)\,\pi(r') = \pi(rr') \tag{4.7.2}$$

where, $\pi : r \longmapsto I + r$. π is a surjective **ring** homomorphism. Also note that the right multiplicative semi-group cosets Ir has **not** anything to do with the quotient ring. If $(R, \cdot)$ is a monoid then one could show that $(R/I, \otimes)$ is a unitary ring, by checking that $I + 1$ is the unity of monoid. Remember, I is a proper ideal of R and hence $1 \notin I$.

Definiteion 4.7.4. *Maximal Ideal : Ideal* I *in a ring* R *is a maximal ideal if for any other ideal* J *in* R *we have* $J \subseteq I \subseteq R$*.*

Definiteion 4.7.5. *Prime Ideal : Ideal* I *in a ring* R *is said to be a prime ideal if it is a proper ideal of* R *and if* $ab \in I$ *implies* $a \in I$ *or* $b \in I$*.*

Definiteion 4.7.6. *Principal Ideal Domain (PID) : An integral domain in which every ideal is a principal ideal is called a principal ideal domain.*

- every subring of a field is a domain.
- for every domain there is a field containing domain as a subring. This containing field is called **fraction field** of the domain.
- a subfield of a ring R is a subring that is a field.

4.8. Modules

Definiteion 4.8.1. *R-Module : An R-Module is a structure built of a duet of sets and a quartet of binary operations,* $\left([\mathbf{M}, R], \left[\vec{+}, \odot, +, \cdot\right]\right)$ *where, the substructure* $\left(\mathbf{M}, \vec{+}\right)$ *is an abelian (commutative) group, and the substructure* $(R, [+, \cdot])$ *is a ring. Additionally,*

(1) $\odot$ *is the action of Abelian group* $(R, \cdot)$ *on the set* $\mathbf{M}$.
(2) $\mathbf{M}$ *is an* R*-set with respect to* $\odot$ *action. That is, as you might remember from the definition (4.3.2) of a* G*-set,* $\forall x \in \mathbf{M}$ *we have* $(r_1 \cdot r_2) \odot x = r_1 \odot (r_2 \odot x)$*;* $\forall r_1$ *and* $\forall r_2 \in R$ *and* $e \odot x = x \; \forall x \in \mathbf{M}$*, where* e *is the unity of* $(R, \cdot)$.
(3) *we have* $(a + b) \odot x = a \odot x \stackrel{\rightarrow}{+} b \odot x$.

Please note that we have differentiated between addition in group $\left(\mathbf{M}, \stackrel{\rightarrow}{+}\right)$ with addition in the ring $(R, [+, \cdot])$. Action $\odot$ maps addition $+$ in the ring to addition $\stackrel{\rightarrow}{+}$ in the group $\mathbf{M}$. You may notice that all properties of a G-set transfers to an R-Module by action $\odot$ of R on $\mathbf{M}$.

Definiteion 4.8.2. *Left* R*-Module : In the previous definition the action* $\odot$ *of* $(R, \cdot)$ *is defined at the* ***left*** *of the set* $\mathbf{M}$. *Hence, it is a* ***left*** R*-Module.*

Normally by an R-Module we mean a left R-Module.

Definiteion 4.8.3. *Right* R*-Module : If the action* $\odot$ *of group* $(R, \cdot)$ *is defined on the* ***right*** *of* $\mathbf{M}$ *then the* R*-Module is a* ***right*** R*-Module.*

Definiteion 4.8.4. *Commutative* R*-Module : An* R*-Module is told to be a commutative* R*-Module when it is both left* R*-Module and right* R*-Module.*

4.9. R-Algebras

Definiteion 4.9.1. *R-Algebra : Assume* R *is a commutative ring then the duet of sets and a quintet array of binary operations,* $\left([\mathbf{M}, R], \left[\bullet, \stackrel{\rightarrow}{+}, \odot, +, \cdot\right]\right)$ *where*

(1) $\left([\mathbf{M}, R], \left[\stackrel{\rightarrow}{+}, \odot, +, \cdot\right]\right)$ *is an* R*-Module and*
(2) $(\mathbf{M}, \bullet)$ *is a commutative semigroup.*
(3) $\left(\mathbf{M}, \left[\stackrel{\rightarrow}{+}, \bullet\right]\right)$ *is a commutative ring and*
(4) $\forall r \in R$ *we have* $(x_1 \bullet x_2) \odot r = x_1 \bullet (x_2 \odot r)$*;* $\forall x_1$ *and* $\forall x_2 \in \mathbf{M}$.

Hence, in an R-Algebra two rings $(R, [+, \cdot])$ and $\left(\mathbf{M}, \left[\stackrel{\rightarrow}{+}, \bullet\right]\right)$ become hinged through the action $\odot$.

Definiteion 4.9.2. *Associative division* R*-Algebra : If* $(\mathbf{M}, \bullet)$ *assumed to be a group rather than a semigroup then* R*-Algebra is called an associative* R*-Algebra.*

Definiteion 4.9.3. *Inner Products in* R*-Module: It is possible to define a binary operation* $\circ : \mathbf{M} \times \mathbf{M} \longrightarrow R$ *with certain properties to create a duet of sets and a quintet array of binary operations,* $\left([\mathbf{M}, R], \left[\circ, \stackrel{\rightarrow}{+}, \odot, +, \cdot\right]\right)$.

(1) $\left([\mathbf{M}, R], \left[\vec{+}, \odot, +, \cdot\right]\right)$ *is an* R*-Module and*
(2) $\circ$ *is commutative.*
(3) $\circ$ *is distributive over* $\vec{+}$*.*
(4) $\forall r \in R$ *we have* $(x_1 \circ x_2) \odot r = x_1 \circ (x_2 \odot r)$*;* $\forall x_1$ *and* $\forall x_2 \in \mathbf{M}$*.*

We should distinguish carefully between R-Algebras and Inner Products in R-Modules, though they have an R-Module as a common part. The inner product in R-Module structure has not those nice algebraic properties of an R-Algebra, in being two hinged rings through the action of a group. For example, $(\mathbf{M}, \circ)$ is **not** a semigroup and $\left(\mathbf{M}, \left[\vec{+}, \circ\right]\right)$ is **not** a ring. Nevertheless it become more important when we create the similar structure of an inner product space on vector spaces later. We also should always use modifier R- to distinguish it with a similar structure, later we build on fields, where we use modifier K- to contrast it.

4.10. Fields

Definiteion 4.10.1. *Field : A field is a division ring* $(R, [+, \cdot])$ *where* $(R, \cdot)$ *is an abelian (commutative) group.*

In a field both addition and multiplication are commutative. It is customary to use the letter K as the set in the field in place of R and leave the letter R to be used only when a ring is involved. Hence, we show a field as $(K, [+, \cdot])$.

4.11. Vector Spaces

Definiteion 4.11.1. *K-Vector Space : A K-Vector Space is a structure built of a duet of sets and a quartet of binary operations,* $\left([\mathbf{V}, K], \left[\vec{+}, \odot, +, \cdot\right]\right)$ *where, the substructure* $\left(\mathbf{V}, \vec{+}\right)$ *is an Abelian (commutative) group, and the substructure* $(K, [+, \cdot])$ *is a field. Additionally,*

(1) $\odot$ *is the action of Abelian group* $(K, \cdot)$ *on the set* $\mathbf{V}$*.*
(2) $\mathbf{V}$ *is a K-set with respect to* $\odot$ *action. That is, as you might remember from the definition (4.3.2) of a G-set,* $\forall \mathbf{v} \in \mathbf{V}$ *we have* $(r_1 \cdot r_2) \odot \mathbf{v} = r_1 \odot (r_2 \odot \mathbf{v})$*;* $\forall r_1$ *and* $\forall r_2 \in K$ *and* $e \odot \mathbf{v} = \mathbf{v}$*,* $\forall \mathbf{v} \in \mathbf{V}$*, where e is the unity of* $(K, \cdot)$*.*
(3) *we have* $(r_1 + r_2) \odot \mathbf{v} = r_1 \odot \mathbf{v} \vec{+} r_2 \odot \mathbf{v}$*.*

Note that we have differentiated between addition in group $\left(\mathbf{V}, \vec{+}\right)$ with addition in the field $(K, [+, \cdot])$. Action $\odot$ maps addition $+$ in the field to addition $\vec{+}$ in the group $\mathbf{V}$.
You may notice that all properties of a G-set transfers to a K-Vector Space by action $\odot$ of K on $\mathbf{V}$.
$(K, [+, \cdot])$ is called the set of **scalars** for the K-Vector Space.

DEFINITEION 4.11.2. *Functional : A mapping f from $K-$vector space $\mathbf{V}$ to its underlaying field K, i.e., $f : \mathbf{V} \longrightarrow K$ is called a functional.*

Addition of vectors in a vector space is well-defined by the group structure $\left(\mathbf{V}, \vec{+}\right)$. Therefore we can follow to this definition.

DEFINITEION 4.11.3. *Linear Combination : Assume that we have n vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ in $K-$vector space $\mathbf{V}$, and scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$ selected from an indexed family $\{\alpha_i\}_{i\in J}$ of scalars in K. We can build a vector $\mathbf{v}$ using definition 4.11.1, as*

$$\mathbf{v} = (\alpha_1 \odot \mathbf{v}_1) \vec{+} (\alpha_2 \odot \mathbf{v}_2) \vec{+} \cdots \vec{+} (\alpha_n \odot \mathbf{v}_n)$$

Or, when there is no ambiguity in the binary operations involved, we can simply write this as

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$$

*We call this **a** linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$.*

Alternatively we can decompose $\mathbf{v}$ into vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$.

DEFINITEION 4.11.4. *Decomposing a Vector : Assume we have vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ taken from the index family $\{\mathbf{v}_i\}_{i\in J}$ of vectors and the vector $\mathbf{v}$ in $K-$vector space $\mathbf{V}$, then there exist scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that,*

$$\mathbf{v} = (\alpha_1 \odot \mathbf{v}_1) \vec{+} (\alpha_2 \odot \mathbf{v}_2) \vec{+} \cdots \vec{+} (\alpha_n \odot \mathbf{v}_n)$$

Or, when there is no ambiguity in the binary operations involved, we can simply write this as

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$$

Or, even more compactly as,

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i\mathbf{v}_i$$

*We call this **a** decomposition of vector $\mathbf{v}$ or resolving vector $\mathbf{v}$ into the components $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$.*

We show the decomposition as a column,

$$\mathbf{v} = \begin{pmatrix} \alpha_1\mathbf{v}_1 \\ \alpha_2\mathbf{v}_2 \\ \vdots \\ \alpha_n\mathbf{v}_n \end{pmatrix}$$

Then the related co-vector can be shown as $b^* = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and sometimes is called a row vector.
K^n defined so is a vector space by its own but in the context of vector space $\left([\mathbf{V}, K], \left[\vec{+}, \odot, +, \cdot\right]\right)$ the $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is the corresponding co-vector of $\mathbf{v}$

DEFINITEION 4.11.5. *Span of a Vector Space : An index family $\{\mathbf{v}_i\}_{i\in J}$ of vectors in $K-$vector space $\mathbf{V}$, is said to span $\mathbf{V}$ if each vector $\mathbf{v} \in \mathbf{V}$ can be decomposed or resolved into the finite number of vectors selected from the family of vectors $\{\mathbf{v}_i\}_{i\in J}$.*

As $\left(\mathbf{V}, \vec{+}\right)$ is an Abelian group so we have a $\mathbf{0}$ vector. Let us decompose this $\mathbf{0}$ vector.

DEFINITEION 4.11.6. *Linear Independence of Vectors : Assume, for the $\mathbf{0}$ vector, we have a decomposition $\mathbf{0} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$. Then we say the set of vectors $(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n)$ taken from the index family $\{\mathbf{v}_i\}_{i\in J}$ of vectors in $K-$vector space $\mathbf{V}$ are linearly independent if $\alpha_i = 0$ for all $i = 1, 2, \cdots, n$.*

DEFINITEION 4.11.7. *Basis of a Vector Space : A set of vectors $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n)$ taken from an index family $\{\mathbf{v}_i\}_{i\in J}$ of vectors in $K-$vector space $\mathbf{V}$, is said to be a basis for $\mathbf{V}$ if they are linearly independent and $\mathcal{B}$ spans $\mathbf{V}$.*

DEFINITEION 4.11.8. *Finite Dimensional Vector Space : If the basis set of vectors $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n)$ are taken from a finite index family $\{\mathbf{v}_i\}_{i\in J}$ of vectors in $K-$vector space $\mathbf{V}$, then we say that the vector space $\mathbf{V}$ is a finite dimensional vector space.*

Later we are going to use the vector spaces in different contexts such as a Hilbert space or a Banach space. In a topological context the idea of dimension will be revisited in its own way.

4.12. K-Algebras

DEFINITEION 4.12.1. *K-Algebra : Assume K is a field then the duet of sets and a quintet array of binary operations, $\left([\mathbf{V}, K], \left[\bullet, \vec{+}, \odot, +, \cdot\right]\right)$ where*

1. $\left([\mathbf{V}, K], \left[\vec{+}, \odot, +, \cdot\right]\right)$ *is a K-vector space and*
2. $(\mathbf{V}, \bullet)$ *is a commutative semigroup.*
3. $\left(\mathbf{V}, \left[\vec{+}, \bullet\right]\right)$ *is a field and*
4. $\forall r \in K$ *we have* $(x_1 \bullet x_2) \odot r = x_1 \bullet (x_2 \odot r)$*;* $\forall x_1$ *and* $\forall x_2 \in \mathbf{V}$.

Hence, in a K-Algebra two fields $(K, [+, \cdot])$ and $\left(\mathbf{V}, \left[\vec{+}, \bullet\right]\right)$ become hinged through the action $\odot$.

DEFINITEION 4.12.2. *Associative division K-Algebra : If $(\mathbf{V}, \bullet)$ assumed to be a group rather than a semigroup then K-Algebra is called an associative K-Algebra.*

DEFINITEION 4.12.3. *Inner Products in K-Vector Space : It is possible to define a binary operation $\circ : \mathbf{V} \times \mathbf{V} \longrightarrow K$ with certain properties to create a duet of sets and a quintet array of binary operations, $\left([\mathbf{V}, K], \left[\circ, \vec{+}, \odot, +, \cdot\right]\right)$.*

(1) $\left([\mathbf{V}, K], \left[\vec{+}, \odot, +, \cdot\right]\right)$ *is a K-vector space and*
(2) $\circ$ *is commutative.*
(3) $\circ$ *is distributive over* $\vec{+}$.
(4) $\forall r \in K$ *we have* $(x_1 \circ x_2) \odot r = x_1 \circ (x_2 \odot r)$; $\forall x_1$ *and* $\forall x_2 \in \mathbf{V}$.

CHAPTER 5

A Hint on Category and Universal Algebra

5.1. Morphism and Categories

DEFINITEION 5.1.1. *Categorical Epimorphism : We say a mapping f is categorically an* ***epimorphism*** *if for an arbitrary pair of mappings g and h the equality $gf = hf$ always implies $g = h$.*

It is also said that in an epimorphism the composition of mappings satisfies the **right cancellation law**. You can prove that a mapping $f : X \longrightarrow Y$ is a surjection if and only if it is an epimorphism.

DEFINITEION 5.1.2. *Categorical Monomorphism : We say a mapping f is categorically a* ***monomorphism*** *if for an arbitrary pair of mappings g and h the equality $g = h$ always implies $fg = fh$.*

It is also said that in a monomorphism the composition of mappings satisfies the **left cancellation law**. You can prove that a mapping $f : X \longrightarrow Y$ is an injection if and only if it is a monomorphism.

5.2. Products

5.3. Universal Algebra

DEFINITEION 5.3.1. *Alphabet : Any set of symbols is called an alphabet. An element of an alphabet, hence, is a symbol.*

DEFINITEION 5.3.2. *Bag : This is a set that multiple occurrence of an element, in contrast to sets, is important but similar to sets, their order of occurrence is not. We use straight brackets to show bags.*

Therefore, bags $[a, b, c, a]$ and $[a, b, c]$ are **not** equal but bags $[a, b, c, a]$ and $[a, b, a, c]$ are equal.

Definiteion 5.3.3. *List : List is a bag where order is also important. We use parentheses to show a list. A list is actually an $n-$tuple. An empty list is shown by* ( ) *or* Λ.

In different contexts, a **finite** list is called a string or a word.
Therefore, lists (a, b, c, a) and (b, a, c, a) are **not** equal.
Lists are shown in parentheses and a separator comma between symbols. Strings and words are shown with commas removed from the list; hence $(baca)$ is a string or a word.
The leftmost element of the strings (lists, words) is called head of the string. Remaining part is the tail.

Definiteion 5.3.4. *Language : A language is a set of strings.*

Definiteion 5.3.5. *Grammar : A grammar is a set of rules to create new strings from the alphabet of a language or from the existing strings of a language on the condition of using only finite steps. Two important rule of any language is concatenation (join, juxtapositioning, paste) of strings and breaking (dividing, cutting) a string.*

Any concatenation and breaking of strings can be exhausted to a sequence of join and cut of a single head element.

Definiteion 5.3.6. *$n-$ary Operation : An $n-$ary operation on X is a mapping ${}_nf : X^n \longrightarrow X$. n is said to be the arity of ${}_nf$.*

Remember from that X^0 has the empty set as its only element. Therefore, a mapping from X^0 to X is a nullary operation and gives $f(\varnothing)$, that is, a constant in the co-domain X.
Similarly, a unary operation is a mapping from X^1 to X. Remember X^1 and X are different.
One can make an alphabet of $n-$ary operations; that is, a set of certain number of $n-$ary operations ${}_nf$. For example, $\mathcal{F} = \{{}_0f, {}_1g, {}_2f, {}_1h, {}_2g, {}_1h, {}_0g, {}_2h, \cdots\}$. Such a set is called a language of **algebras**. This word, **algebra**, has used with different meanings in different contexts and is very confusing. We tried to give every usage of this notion in this book, clearly cut in its own context. $\mathcal{F}$ has a subset of all $n-$ary mappings shown by $\mathcal{F}_n$. Therefore, for instance, $\mathcal{F}_2 = \{{}_2f, {}_2g, {}_2h, \cdots\}$.
Any element ${}_nf \in \mathcal{F}_n$ where $n \geq 2$, but is finite, can be expressed as composition of binary operation but it is not in the scope of this book to prove that. Therefore we never encounter with operations more than a binary operation.

Definiteion 5.3.7. *Algebra of type $\mathcal{F}$: Assume we have taken a set of mapping symbols from $\mathcal{F}$ to define a family F of operations on a set A with different arities then we call the duet $[A, F]$ an algebra $\mathbf{A}$ of type $\mathcal{F}$. A is called the underlying set of the algebra $[A, F]$.*

To emphasise f is an operation on a certain set A, we use the notation $f^{\mathbf{A}}$ when the context requires.

CHAPTER 6

Symmetry and Transformations

6.1. Symmetric Groups

DEFINITEION 6.1.1. *Symmetric Monoid M_X : Assume X is a set. The set of all mappings on X is said to be the symmetric monoid on X. Composition of mappings assumed as the binary operation that reveals the monoid structure.*

Here the identity mapping i_X is the unity member of the monoid.

DEFINITEION 6.1.2. *Symmetric Group S_X : Assume X is a set. Then the set of all **bijections** on X forms (with composition operator) a group called the symmetric group on X, shown as S_X. An element of S_X is called a **permutation**.*

Remember later to distinguish **symmetric group** with the **group of symmetries** of a shape, though they are related as will be seen.

DEFINITEION 6.1.3. *Symmetric group of degree n : Let set $X = \{1, 2, 3, \cdots, n\}$, then the symmetric group is said to have the degree n. We show this group by S_n.*

One might show that number of elements of the S_n is equal to $n!$. That is $|S_n| = n!$. Each member of S_n is one permutation of set X.
Hence, we have a set of natural numbers sorted consecutively from 1 to n, and a permutation shuffles them in a set of number with another arrangement. For example 1 is not at its original place as the first member. Now it is in fourth place. 2 is at first place. 3 is at seventh place. 4 is at second place. 5 just happens to be still at fifth place and so on. We see that jugstapositioning in Figure 6.1.

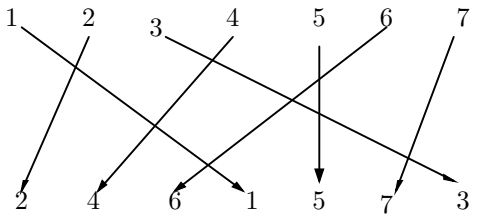


FIGURE 6.1. A permutation on $X = \{1, 2, 3, 4, 5, 6, 7\}$.

Remark 6.1.1. Notation: A permutation is shown by a row such as $(2, 4, 6, 1, 5, 7, 3)$, and frequently without commas separating the elements like, $(2\,4\,6\,1\,5\,7\,3)$.

Definiteion 6.1.4. *Inversion in a Permutation : In the resulting jugstaposition of a permutation each occurrence of a number smaller than the first number in the row is called an inversion. In the example of Figure 6.1, 1 is the only inversion.*

In the set X, before permuting its elements to a new arrangement, numbers are ordered consecutively upward. We are interested to know how each number has become far from its original position and if we want to have the original order back by moving each entry one position how many movements we should do.

Definiteion 6.1.5. *Number of Inversion : We should add all the inversions of the arrangement by taking the first entry until remains no inversion. Adding the partial number of inversions together gives the total number of inversions. We show this number with J.*

Example 6.1.1. *(Please refer to Figure 6.1) :*

$$\begin{aligned} 2, 4, 6, 1, 5, 7, 3 &= 1 \\ 4, 6, 1, 5, 7, 3 &= 2 \\ 6, 1, 5, 7, 3 &= 3 \\ 1, 5, 7, 3 &= 0 \\ 5, 7, 3 &= 1 \\ 7, 3 &= 1 \end{aligned}$$

Hence, $J = 1 + 2 + 3 + 0 + 1 + 1 = 8$

If you look at the figure 6.1 in the above example J is equal to number of intersections of arrows with each other. This is the easiest way of calculating J. A permutation could be an even permutation or an odd permutation.

Definiteion 6.1.6. *Even Permutation : In an even permutation number of inversions J is an even number, similar to previous example 6.1.1.*

Definiteion 6.1.7. *Odd Permutation : In an odd permutation number of inversions J is an odd number.*

In Figure 6.1 have a look at the subset $(1, 2, 4)$. They loop and end without any other number comes into the permutation, like this $1 \to 2$; $2 \to 4$; $4 \to 1$. Similarly, we have the subset $(3, 6, 7)$, where $3 \to 6$; $6 \to 7$; $7 \to 3$. These permutations are called factors of original permutation $(2, 4, 6, 1, 5, 7, 3)$. We also note the singleton factor (5). We write, as a convention, $(2, 4, 6, 1, 5, 7, 3) = (1, 2, 4)(3, 6, 7)(5)$.

Definiteion 6.1.8. *Alternating Group A_n : All even permutation of S_n form a subgroup of S_n that is called the alternating group.*

One easily can show that this is an abelian subgroup. Perhaps the reader can remember from chapter one that when we have a set X we can make another set related to that set by gathering all mappings defined on that set. We called that set a pre-set that can be shown as ${}^{X}X$. We see that $S_n \subseteq {}^{X}X$. We can be choosier in selecting a certain subset of all the possible mappings on a set. For instance, all linear mappings, or all mappings with the co-domain not the set X, but some other set, for example all the mappings from set X to set of real numbers $\mathbb{R}$. Here, in defining symmetric groups, we selected all the bijections on the set X among all the possible mappings. Then, the resulting set of mappings shows notable algebraic structures of interest such as a group or ring or a vector space.

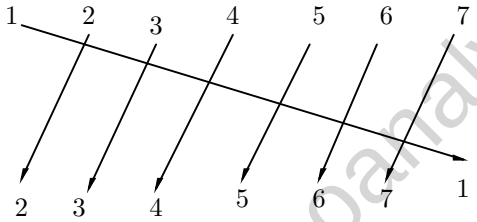


FIGURE 6.2. A cyclic permutation on $X = \{1, 2, 3, 4, 5, 6, 7\}$.

DEFINITEION 6.1.9. *Cyclic Permutation : In cyclic permutation each element of the row maps into the next member. The last member maps into the first element. See Figure 6.2 on page 63.*

Instead of referring, let me repeat few definitions from here, 4.3.

DEFINITEION 6.1.10. *Action of a Group on a Set: Assume there is a binary operation, shown with symbol $\odot$, such that $\odot : G \times X \longrightarrow X$. Mapping $\odot$ is called the action of G on set X.*

DEFINITEION 6.1.11. *G-set of a Set : Let $\odot$ be an action of group G on set X. We say X is the G-set if $\forall x \in X$ we define*

1. *$e \odot x = x$ where e is unity in G.*
2. *$(g_1 * g_2) \odot x = g_1 \odot (g_2 \odot x)$; $\forall g_1$ and $\forall g_2 \in G$, where $*$ is the binary operation in G.*

*Elements of G are also called **operators** on X when the context requires that.*

Note that in both definitions above, we took two members from G and X respectively in that order and G is on the **left** of X. The result is an element in X. We equally could define an action of G on the **right** of X.

Example 6.1.2. *: Assume $\mathbb{Z}_2$ with a group structure acting on the sphere S^1. Define the action of $[0]$ as the neutral action and $[1]$ as half a circle rotation. The S^1 is a $\mathbb{Z}_2$−set. Similarly, action of group $\mathbb{Z}_n$ on S^1 can be defined, where $[m]$ gives a $2m\pi/n$ rotation to any $x \in S^1$ for $m \in [m]$.*

DEFINITEION 6.1.12. *X_g-set : Let $\odot$ be an action of group G on set X. A subset Γ of X is said to be X_g-set if $\forall x \in \Gamma$ we have $g \odot x = x$ where g is an element in G.*

For example, a G-set X is an X_e-set.

DEFINITEION 6.1.13. *G_X-group (Stabilizer): Let $\odot$ be an action of group G on set X. A subset H of G is said to be G_X-set or **isotropic** subgroup or **stabiliser** of G if $\forall g \in H$ we have $g \odot x = x$ where x is an element in X.*

G_X-group is a subset (sub**group**) of G, but X_g-set is a sub**set** of X.

DEFINITEION 6.1.14. *Transformation : Let Γ be any set. Then a bijection g^* of Γ to Γ, that is, $g^* : \Gamma \xrightarrow{bijection} \Gamma$ is called a transformation of Γ.*

DEFINITEION 6.1.15. *Invariants of a Transformation : In transformation $g : \Gamma \longrightarrow \Gamma$ those points $x \in \Gamma$ where $g(x) = x$, that is, points that their images are themselves are called invariant points of transformation.*

DEFINITEION 6.1.16. *Group G^* of Transformations of Γ(Transformation group acting on Γ) Narrow Definiteion : Assume G^* is a set of transformations on a set Γ. If this set includes a neutral (identity) transformation and along each transformation it also includes its inverse transformation then one can recognize a group structure in set G^*. Then set G^* is said to be **a** group of transformations of Γ in the narrow sense.*

DEFINITEION 6.1.17. *Transitive Group G^* of Transformations of Γ Narrow Definiteion : Assume G^* is a transformation group acting on Γ. If there is a $g \in G^*$ such that for any pair of elements $\xi \in \Gamma$ and $\zeta \in \Gamma$ we have $g(\xi) = \zeta$, then group G^* is called a transitive group acting on Γ.*

DEFINITEION 6.1.18. *Group G^* of Transformations of Γ(Transformation group acting on Γ)Narrow Definiteion : This is the set of **all** transformations on the set Γ.*

The group G^* of all transformations on Γ is a transitive group.

DEFINITEION 6.1.19. *Group G of Transformations of Γ(Transformation group acting on Γ)Wider Definiteion : Assume G is a group, X is a **set** of transformations on Γ (not necessarily a group). Then G is said to be a transformation group acting on Γ in a wider sense if $\forall g \in G$ there is $x^* \in X$ such that $x^* = \tau(g)$, where τ is a mapping from G into X with the property that $\tau(gh) = \tau(g)\tau(h)$.*

Remark 6.1.2. Reader notices that gh is actually $g \bullet h$ a group operation, while $\tau(g)\tau(h)$ is composition of two transformations $\tau(g) \circ \tau(h)$ in X, a set which has not any group structure yet, but is to be structured by group structure of G. (N.B., $\xi = \tau(g)$ is one transformation in X and $\zeta = \tau(h)$ is another transformation in X different from ξ, but both acting on Γ as a common domain of theirs, say, $\beta = \zeta(\alpha)$ and $\gamma = \xi(\beta)$; hence, $\xi(\zeta(\alpha)) = \gamma$, where $\alpha, \beta, \gamma \in \Gamma$; that is, $\xi \circ \zeta$) Don't confuse them as being the same element, as they both are images under τ. All these elaborations are to reach to that goal; giving X structure of a group.

Condition imposed on the τ guarantees (please have a try to prove[1]) that $G^* = \tau(G) \subseteq X$ has got a neutral transformation of Γ and an inverse of each transformation along with some transformations; hence, a minimum core of a group G^* of transformations in its narrow sense, exists among the elements of X. By corresponding G through mapping τ we establish a homomorphism from G to this group. Then group G is a group of transformation acting on Γ in its wider definition. And the image $G^* = \tau(G)$ is the core of X that makes a group of transformation in the narrow sense. Therefore, now $\tau : G \longrightarrow G^*$ is a surjection. We know that in nature there is no structure. There are no rotation no reflection and no symmetry. Still we have abstract group structures or vector spaces or algebras which we like to recognize them among shapeless nature, to make our interpretation of nature easier or to make us able to utilize a natural phenomenon. We have done it from ancient Egyptian, Babylonian and at the crest of them Greek mathematical abstractions to present time. Groups of transformations are very useful tools for study of nature, and group theory, actually started from the study of group of transformations. For this reason we followed Pontryagin [**26**] to distinguish between a narrow sense and a wider definition. This takes us to define a group of transformations acting on a set up to an isomorphism.

Example 6.1.3. *Antipodal map transformation : The mapping that sends a point on the circumference of circle (generally, any sphere S^n) to its diagonal opposite point is called an antipodal mapping. Among transformations of the plane we recognise this as $\xi(x) = -x$, where $x \in S^1$. Applying this transformation twice you'll be back to the same point $\xi(\xi(x)) = x$ or $\xi(\xi(x)) = \epsilon(x)$, the identity mapping. In language of group theory we have a subset of all transformations on plane which can be identified as a group, $G^* = \{\xi, \epsilon\}$. Thus $g^* = g^{*-1}, g^{*2} = e, \forall g^* \in G^*$. On the other hand, we have group $G = \{[0], [1]\}$ acting in wider sense on S^n which is isomorphic to G^* that acts on a narrow sense. We always prefer to use group G.*

Therefore, it is more convenient to use groups such as $\mathbb{Z}_n$ when talking about rotation and such things on the plane rather than $\xi(x)$ and $\xi(\xi(x))$, and so on. You already might be puzzled what was about a set (recognisable as group) such as $G = \{[0], [1]\}$ to rotation and such.

Definiteion 6.1.20. *Kernel of Ineffectiveness of Group G of Transformations of Γ : Kernel of surjection τ is called the kernel of ineffectiveness of G.*

In this terminology, it is ineffective in the sense that all the corresponding members of G that map into the neutral transformation in G^* come together in the form of the kernel. They act as identity on Γ.

Definiteion 6.1.21. *Effective Transformation Group : If τ is an isomorphism, that is, it is a bijection then G is an effective transformation group*

In this case the kernel of the injection is the single identity transformation in G. Now G can be identified with G^* and elements of G can be regarded as transformations of Γ.

Definiteion 6.1.22. *Group G of Transformations of Γ (Transformation group acting on Γ) Wider Definiteion : If the set X contains all transformations on the*

[1]For example, let the neutral element $e \in G$ then for $g \in G$, $x^* = \tau(g) = \tau(e \bullet g) = \tau(e) \circ \tau(g) = \tau(e) \circ x^*$; hence, $\tau(e)$ should be a neutral element in G^* and we show it as $e^* \triangleq \tau(e)$.

set Γ *then the group* G *defined in 6.1.19 is* ***the*** *group of transformations on* Γ *in a wider sense.*

Frequently we show the group G of transformations acting on a set Γ as (G, Γ).

DEFINITEION 6.1.23. *Transitive Group* G *of Transformations of* Γ*, Wider Definiteion: Assume* G *is a transformation group acting on* Γ *in the wider sense. If* $G^* = \tau(G)$ *is a transitive group in narrower sense then group* G *is called a transitive group acting on* Γ *in wider sense.*

The group G of all transformations on Γ is a transitive group.

DEFINITEION 6.1.24. (G, Γ) *similar to* (G', Γ') *: A pair of mappings* (f, ϕ) *is said to be a* ***similarity*** *between* (G, Γ) *and* (G', Γ') *if* $f : G \xrightarrow{iso} G'$ *is an isomorphism and* $\phi : \Gamma \xrightarrow{inj} \Gamma'$ *is an injection. In this case, the pairs* (G, Γ) *and* (G', Γ') *are called* ***similar****.*

DEFINITEION 6.1.25. *Orbit of* x *under* σ *: Assume* X *is a set and* $x \in X$*. Also, let* σ *be a permutation in symmetric group* S_X *of permutations of* X*. Then we define the orbit of* x *under* σ *as the set of all* $x\sigma^n$ *where* $n \in \mathbb{Z}$*. We show this orbit as* $O_{x,\sigma}$*.*

Orbit should bring in mind the notion of a coset. We have a set of transformations $\{\sigma, \sigma^2, \sigma^3, \cdots, \sigma^n\}$ mapping (permuting) x once and then twice and so on. Also consider the notion of a submodulus mapping. It is good if the last mapping spits out the x back. But it is not necessary from the definition. Orbit is the track of an element x inside the set X traveling with or riding on a certain mapping σ. Also note that we usually show a mapping of an element x of a set by something like $f(x)$, mapping on the left side and the element on the right side. Frequently, in algebra, we do not use parenthesis and show the mapping on the right side and the element on the left side, such as xf. More generally we can have,

DEFINITEION 6.1.26. *Orbit of* ξ *under* G*: Assume* G *is a group acting by* $\odot$ *on set* Γ*. Then orbit of each* $\xi \in \Gamma$ *is the set of all* $\zeta \in \Gamma$ *defined as the equivalent class* $[\xi]$ *such that,* $[\xi] = \{\zeta \,|\, \zeta \sim \xi;\ \zeta = \xi \odot g;\ \forall g \in G\}$*. Orbit* $[\xi]$ *is called fiber of* ξ*, too. See, Figure 6.3.*

It is all the images of ξ under the operators $g \in G$. Note that, as usual, we show action of group $\odot$ from right on its left side; hence $\xi \odot G$ is the same as $[\xi]$. (Note, let $\zeta_1 = \xi \odot g_1$ and $\zeta_2 = \xi \odot g_2$ Then $\zeta_1 \sim \zeta_2$ and $\zeta_1 \in [\xi]$ and $\zeta_2 \in [\xi]$.)

DEFINITEION 6.1.27. *Orbit Set or Quotient Set of* Γ *by Group* G*: We define the quotient* $\Gamma/[\xi]$ *(partitioning of* Γ *by group* G*) as the orbit set of* Γ *and show it by,* Γ/G*, though the last notation could be confusing (Please, see also 2.6.4). See, Figure 6.4.*

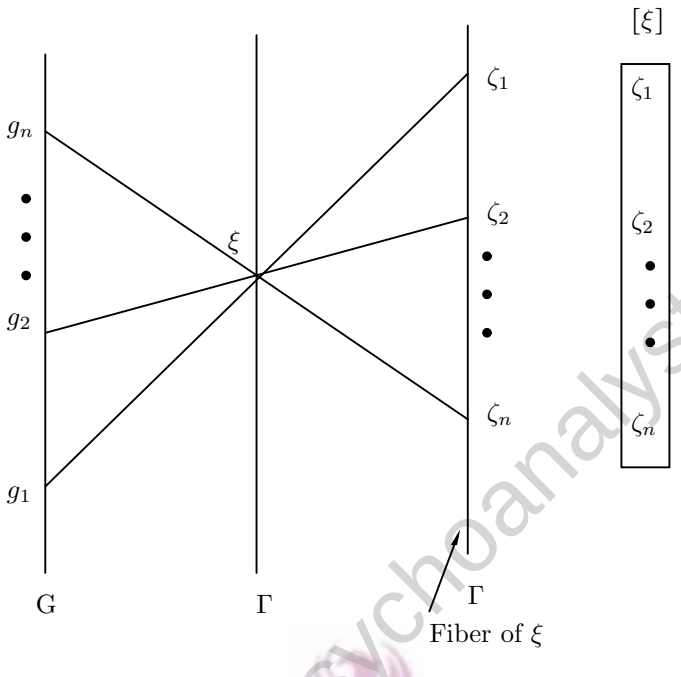


FIGURE 6.3. Fiber of ξ under (or with respect to) group G. In this case is (but not limited to), say, $\{\zeta_1, \zeta_2, \cdots, \zeta_n\}$.

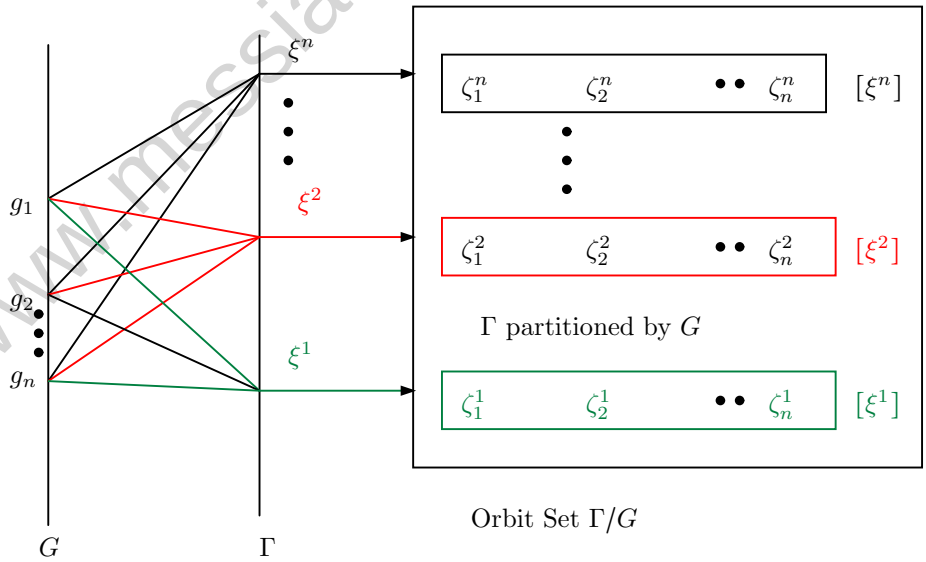
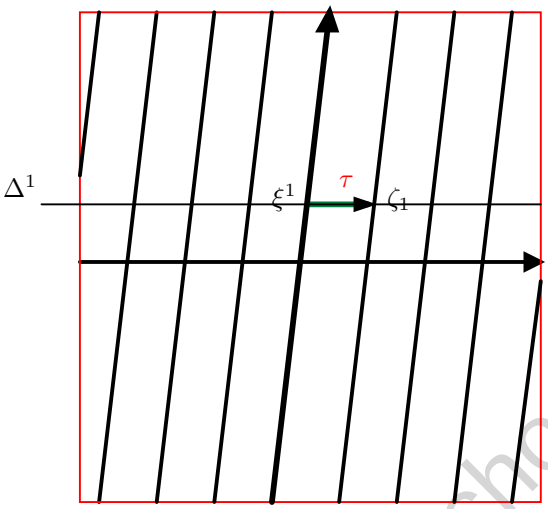


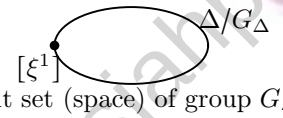
FIGURE 6.4. Partitioning of Γ by group G

Example 6.1.4. *Torus as orbit set of group of translation of plane : See, Figure 6.5. In 6.6 second generator of the torus is shown as σ, acting on the lines of*

the plane in a direction not parallel to lines Δ^k. The resulting torus is not drawn; only cylinder of horizontal partitioning is depicted.



Group G_Δ of horizontal translations:
Group generator is the translation τ.

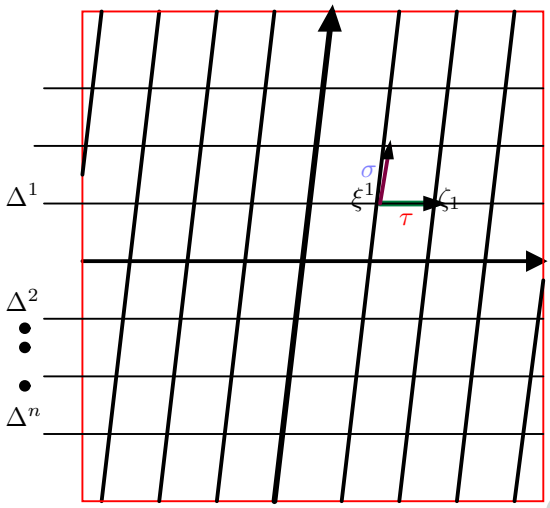


Orbit set (space) of group G_Δ acting on line Δ^1 is a circle.

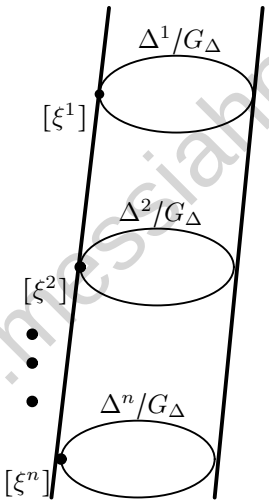
FIGURE 6.5. Partitioning of the line Δ by group G of horizontal translations of the line.

Example 6.1.5. *Klein Bottle as orbit set of group of glide transformations of plane : Please have a look at Figure 6.8. Generators of the bottle are shown as σ and τ, acting on the so-called* ***fundamental region*** *of the plane depicted as a diamond with black lines. Each generator is a glide reflection acting on zigzag lines of the plane. Each generator creates a Mobius strip if acting alone. Compare it with translation that creates a cylinder. Two mobius strips combined create an orbit set as a Klein bottle. Two narrow red doted lines are reflections of ξ^1 and the other edge of the diamond.The bottle is not drawn, but corresponding points of transformations are shown with similar letters. Diamond transformed to a cross with dotted red and green line segments. To put line ξ^1 on its corresponding line ζ_1, point (a) translates and turns until it gets to its corresponding point (a) on the tip of the cross. Every other set of criss-crossed lines of plane transforms to one Mobius strip. If you are good in the drawing you notice that beginning with a directed circle you end up with a circle inverted in direction to the first one but fully adjusted on it. It is a Klein bottle (Armstrong* [**4**]*).*

Example 6.1.6. *Sphere as orbit set of group of half-turn transformations of line segments of plane : See, Figure 6.10. Fundamental region of plane of transformation is repeating triangles (Figure 6.9)that fills the plane. Each half-turn transform*



Group G_Δ of horizontal translations
Group generator translation τ



Orbit set (space) of group G_Δ acting on horizontal lines of plane is a cylinder.

FIGURE 6.6. Partitioning of the plane by group G of horizontal translations of horizontal lines.

alone acting on plane has a cylinder as the orbit. Each line bearing the shown half-turn is identified with a circle in a plane different with other two. Three non-parallel planes define a sphere in the same way as three mutually intersecting lines define a circle.

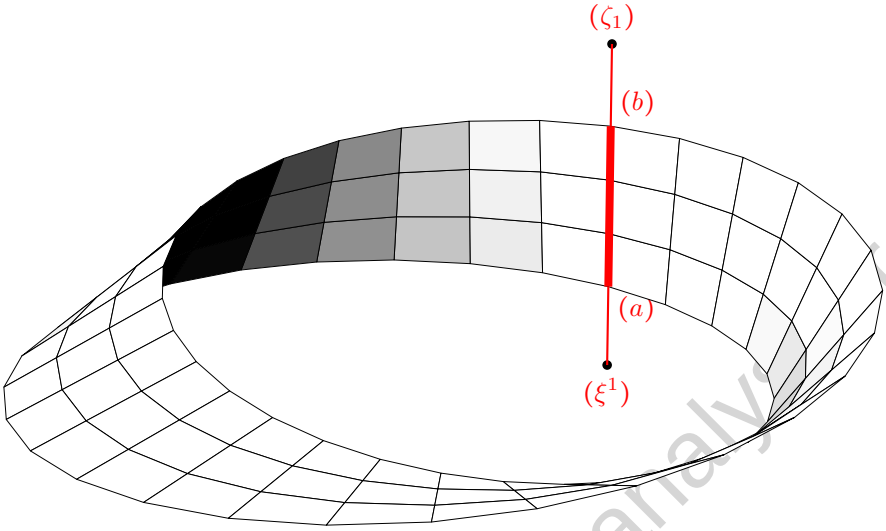


FIGURE 6.7. One generator of group acts on the plane.

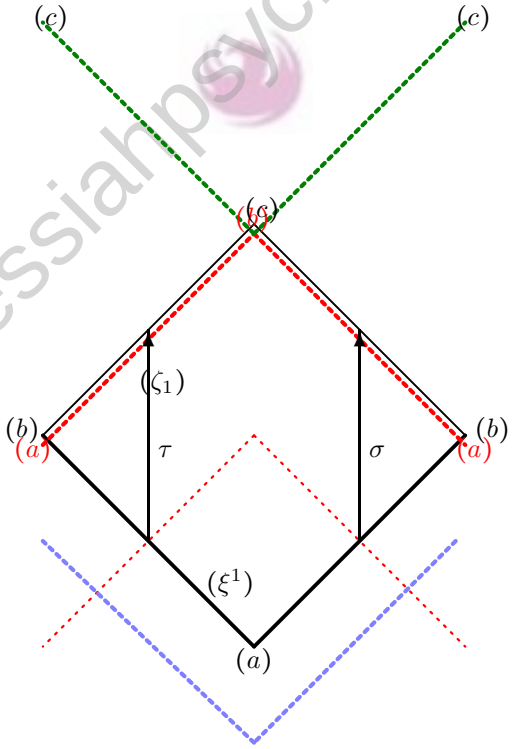


FIGURE 6.8. Partitioning of the plane Γ by group G of glide transformation of plane.

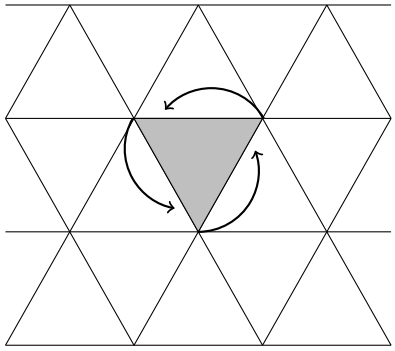


FIGURE 6.9. Half-turn transformations of plane.

Partitioning of the line Δ by group G of horizontal translations of the line

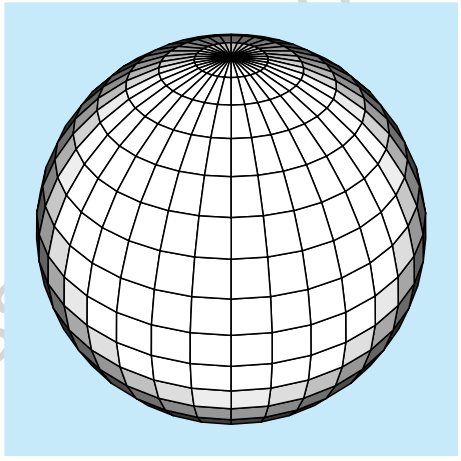


FIGURE 6.10. Sphere as orbit set of half-turn transformations.

DEFINITEION 6.1.28. *Stabilizer H_ξ (shallow) : Assume G is a group acting by $\odot$ on set Γ. Then, we have $H_\xi = \{g \in G | \xi \odot g = \xi\}$. See, Figure 6.11.*

DEFINITEION 6.1.29. *Stabilizer H_ξ (deep) : Let G be a transitive group of transformations acting in a wide sense on set Γ. Assume $g^* \in G^*$ is the associated transformation of $g \in G$ such that, $g^*(\xi) = \zeta$; $\xi, \zeta \in \Gamma$, where ξ is fixed in Γ. See, Figure 6.12. We define set $\psi(\xi, \zeta) = \{g \in G | g^*(\xi) = \zeta\}$. Then, stabilizer H_ξ is defined as $\psi(\xi, \xi)$.*

It is clear that H_ξ is a subgroup of G. Also each $\psi(\xi, \zeta)$ is in the form of $g\psi(\xi, \xi)$, for some $g \in G$ (hint: since G is transitive); hence, a left coset of $\psi(\xi, \xi)$.

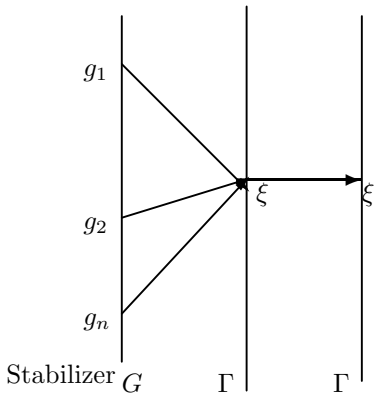


FIGURE 6.11. Stabilizer of ξ under (or with respect to) group G. In this case is (but not limited to), say, $\{g_1, g_2, \cdots, g_n\}$.

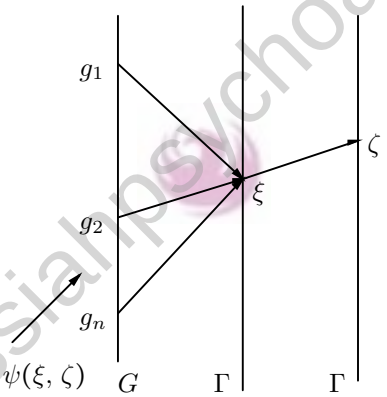


FIGURE 6.12. $\psi(\xi, \zeta)$ under (or with respect to) group G. In this case is (but not limited to), say, $\{g_1, g_2, \cdots, g_n\}$.

Remark 6.1.3. $\psi(\xi, \zeta)$ is a mapping : For each $\xi \in \Gamma$ we have a set of equivalent elements of G partitioned with the corresponding H_ξ. Therefore, we can speak of ψ as a mapping of Γ to G/H_ξ. That is, $\psi : \Gamma \longrightarrow G/H_\xi$, and $\psi : \xi \mapsto H_\xi$. This mapping is a bijection. When there is no risk of ambiguity, we write it just as $\psi(\xi)$.

Assume H is a subgroup of G. Consider action of H, as a group on the G, as a set. Then orbit of $x \in G$ is xH. These orbits for all $x \in G$ create a partition G/H for G.
On the other hand, we have cosets xH that, in their own turn, create a similar partition G/H for G.
It is worthwhile to distinguish these two as one which involves "action" and "orbits" and the other just algebraic cosets.

DEFINITEION 6.1.30. *Affine Space : Assume A is just any set, and V is a vector space over field K. We know, $(V, +)$ is an abelian group. We define an action of this group on the set A by a mapping $+_A : A \times V \longrightarrow A$ such that*

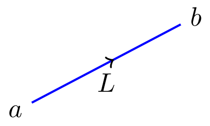


FIGURE 6.13. Oriented Line Segment L.

(1) $a +_A (\mathbf{v} + \mathbf{w}) = (a +_A \mathbf{v}) +_A \mathbf{w}$, *where* $\mathbf{v}, \mathbf{w} \in V$.
(2) $\forall\, a, b \in A$ *there is a unique* $\mathbf{v} \in V$ *such that* $b = a +_A \mathbf{v}$.
(3) a *and* b *are called initial point and terminal point of the vector* $\mathbf{v}$.

We show an affine space built out of the V *by* AV *or* $\mathbb{A}_V$. *When there is no ambiguity we drop* V, *as* $\mathbb{A}$ *or just simply* A.

I have differentiated the affine addition (which is a group action) by putting a subscript A next to it. Most of the time people omit that subtlety, and that is the reason that many do not appreciate the subtle difference of affine space with a vector space. Therefore an affine space is lacking any structure. We give it part of the structure of a vector space through the group underlayer of that vector space. From the before mentioned item (2) we notice that $a = a +_A \mathbf{0}$. Item (2) can interpret the meaning of the group action as **translation** of a point $a \in A$ into another point $b \in A$. Item (3) bounds the *free* vector $\mathbf{v}$ between fixed points a and b, changing a mathematician movable vector, with no intrinsic position, into a physical vector where the point of the insertion of the vector and its direction has a meaning and is important. This new vector object created by the **action** of the group, has also an additional line of **action** that carries the vector (carrying line of the vector). You appreciate that each vector now can be written as an equivalent class of points of AV. By the way, please note that many modern mathematicians are very careful in clarifying if any operation is from the right-side or from the left-side. When there is a danger of ambiguity, I follow that rule, but not when the things are obsessively trivial.
Throughout the algebraic definition of groups, rings, and fields and then definition of modules and vector spaces we have not introduced any notion of order. We did not need to supply these structures with any order relation.

DEFINITEION 6.1.31. *Oriented Line Segments : Configuration* ab, *where* $a, b \in A$, *satisfying item (2) of definition 6.1.30 is said to be an oriented segment in* A. a *is called the initial point and* b *the final point of* ab. *We say* ab *is oriented from* a *to* b. *We call* ab *a line segment, in contrast to vector* $\mathbf{v}$ *and show it as* L.

We can visualise an oriented segment L as a line segment in plane and put an arrow head on the line to further contrast it with a vector in a vector space. Please see Figure 6.13.
An oriented line segment is also called a directed segment and is said to have an orientaion or a sense from a to b.

DEFINITEION 6.1.32. *Labelled Oriented Line Segments : Assume $\mathcal{L}$ is a set of oriented line segments and r is a set called the set of labelles. Then a mapping $l : \mathcal{L} \longrightarrow r$ is said to be a labeling of the line segments in $\mathcal{L}$.*

We are allowed to give the same label to different segments if it is desired.
Now consider a point $c \in A$ such that $c = a +_A s\mathbf{v}$ where $s \in K$. Vector $s\mathbf{v}$ is in the same direction or opposite direction as of the vector $\mathbf{v}$, but scaled. In such a case we say point c is on the line ab. The set of all such points, c, is said to be an oriented line in L. This is the line of **action** of the physical vector $\vec{ab}$. We show this line by (L) or λ. Hence, $\lambda = \{c \in A \,|\, c = a +_A s\mathbf{v},\ \forall s \in K\}$. If things have gone well so far and K be an ordered set, then for $0 \leq s \leq 1$ we say c is a point on the line L, *between* a and b. This transformation imposes an order on points of A. We can show it by writing $a \leq c \leq b$.
Frequently the world is not as complicated and we have $A = K$. In such cases, we can write $c = (1 - s)a + sb$. Note that the affine space is now $\mathbb{K}$ not the K.

DEFINITEION 6.1.33. *Homogeneous Points : Assume $b = a +_A \mathbf{v}$ and $c = a +_A s\mathbf{v}$. Then b and c are said to be homogeneous points on λ. s is called the ratio of homogeneity and could be negative or positive. a is called the origin of homogeneity. If $s = 0$ then a and be b will be the same point of A.*

DEFINITEION 6.1.34. *Reflection Points : Assume $b = a +_A \mathbf{v}$ and $c = a +_A s\mathbf{v}$ and $s = -1$. Then b and c are said to be reflection points. Point a is called the origin of reflection. Nevertheless, these two points are homogeneous with ratio of homogeneity $s = -1$.*

DEFINITEION 6.1.35. *Congruent Segments : Assume $b = a +_A \mathbf{v}$ and $d = c +_A \mathbf{v}$. Then segment ab and segment cd are said to be congruent segments.*

We could start from a set of points and define a vector as the class of all congruent oriented segments identified as one. That is the abstract mathematical free vector in contrast to bound segments. However such approach would be out of fashion.

DEFINITEION 6.1.36. *Rotation of an Oriented Segment : Assume $b = a +_A \mathbf{v}$ and $c = a +_A \mathbf{w}$. Then ca is said to be rotation of ba around the pivotal point a **into** the direction $\mathbf{w}$. Point a is called the invariant point of rotational transformation.*

It can be noted that there exist a vector $\mathbf{u}$ such that $c = b +_A \mathbf{u}$. Then $\mathbf{u}$ is said to be the direction of rotation.

DEFINITEION 6.1.37. *Translation of an Oriented Segment : Assume $b = a +_A \mathbf{v}$ and $d = c +_A \mathbf{v}$ and $d = b +_A \mathbf{w}$. Then segment cd is said to be translation of segment ab in direction of vector $\mathbf{w}$.*

One always can find such a vector for two congruent segment.

Definiteion 6.1.38. *Inversion of an Oriented Segment : Assume* $b = a +_A \mathbf{v}$. *We can assert that* $a = b +_A (-\mathbf{v})$. *Then* ba *is said to be the invert of the* ab. *Interchanging* a *and* b *is called an inversion operation, or a half-turn (around the middle of the segment).*

Please differentiate between inversion and the reflection.

Definiteion 6.1.39. *Convex Subsets of* $\mathbb{A}$ *: A subset* $\mathbb{B}$ *of* $\mathbb{A}$ *is said to be convex if for any pair of points* $a, b \in B$ *we have* $c = (1-s)a + sb \in B$ *for some* $s \in K$.

Definiteion 6.1.40. *Convex Closure of Subsets of* $\mathbb{A}$ *: Assume the subset* U *of* $\mathbb{A}$ *is not convex but there is a subset* V *such that* $U \subset V$ *and* V *is convex. Then the intersection of all such convex subsets* $\overline{V} = \bigcap V$ *is said to be the convex closure of* U. *This is the smallest convex subset containing* U

We have been relax in detailing the definition.

Definiteion 6.1.41. *Identification on a Line Segments : Assume* K *is ordered and* $U = \{s \in K \,|\, 0 \leq s \leq 1\}$ *and there is an order preserving bijection mapping* $\phi : U \longrightarrow L$, *where* L *is the oriented line segment* ab *in* A. *Then we say points of* L *are identified by points in* K. *By this, we mean each point of* L *for any* $L \in A$ *can be recognised with some* $s \in K$. ϕ *is called parametrization of the line segment* L.

L or ab defined in this way is said to be the **unit** segment on the extended line (L) or λ. Extension of $\phi : K \longrightarrow \lambda$ is the parametrization of the line λ.

Definiteion 6.1.42. *Coordinate Map on a Line : Let* $\phi : K \longrightarrow \lambda$ *be a parametrization of* λ *then the inverse of* ϕ *that is* $\phi^{-1} : \lambda \longrightarrow K$ *is said to be a coordinate map or a chart map on* λ.

Coordinate map is also called a *coordinate system* on λ.

Definiteion 6.1.43. *Transition Map : Assume* ψ *is another parametrization of the line* λ, *besides* ϕ, *then for a point* $s \in K$ *we have point* $c = \psi(s) \in \lambda$. *Now for* $c \in \lambda$, *we have* $t = \phi^{-1}(c) \in K$. *Therefore,* $t = \phi^{-1}(\psi(s))$ *maps* $s \mapsto t$, *and the composition map* $\phi^{-1} \circ \psi$ *is a transformation on* K. *Alternatively,* $c \in \lambda$ *we have coordinate* $s = \psi^{-1}(c) \in K$. *Now for* $s \in K$, *we have* $d = \phi(s) \in \lambda$. *Therefore,* $d = \phi(\psi^{-1}(c))$ *maps* $c \mapsto d$, *and the composition map* $\psi \circ \phi^{-1}$ *is a transformation on* λ. *Either of these composition maps is said to be a transition map for* λ

Definiteion 6.1.44. *Substitution Map : Assume* $\lambda \subset \mathbb{A}$, *and let* $c, d \in \lambda$ *any successive chain of bijections that map* $c \mapsto d$ *is called the substitution of* c *by* d.

Definiteion 6.1.45. *Homogeneous (Linear) Transformation : Assume* ab *is an oriented segment such that* $b = a +_A \mathbf{v}$. *Any transformation that maps* $b \mapsto d$ *such that* $d = a +_A \mathbf{w}$, *where* $\mathbf{w}$ *is another vector in* V *is called a linear homogeneous transformation or operator on the affine space* $\mathbb{A}$.

Note that this transformation keeps the initial point of ab fixed but changes the length (the final point) and the direction of ab. Homogeneous transformation rotates and scales the affine vector ab. Having ab parametrized as the unit then ratio of the ad to this unit shown by k is called the ratio of homogeneity (homogeneity).

DEFINITEION 6.1.46. *Affine (Linear) Transformation : Assume* $b = a +_A \mathbf{v}$ *and* $d = e +_A \mathbf{w}$*, where* $\mathbf{v}, \mathbf{w} \in V$ *then a transformation of* ab *to* de *that maps* $a \mapsto d$ *and* $b \mapsto e$ *is called an affine transformation.*

We know there exists a $\mathbf{u} \in V$ such that $d = a +_A \mathbf{u}$. This is the homogeneous transform of ab to ad. Then we have, $a = d +_A (-\mathbf{u})$ which is the segment da; the inversion of ad. Now another homogeneous transformation transforms da to de; (Figure 6.14).

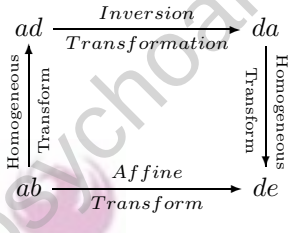


FIGURE 6.14. A commutative diagram for affine transformation.

Assume we are in a three (n) dimension (normed linear) space ($n = 3$) and we have four points ($m = n + 1 = 4$). Select a $k < 3$ ($k < n$), for example $k = 2$. If no three points ($l = k + 1 = 3$) lie on a line, that is, on a subspace with dimension $< k\,(k = 2)$, then these four (m) points are said to be in general positions. Therefore,

DEFINITEION 6.1.47. *General Position : Assume* $n + 1$ *points* $a_1, a_2, \cdots, a_{n+1}$ *in an affine space* $\mathbb{A}$ *with underlying vector space of dimension* n *are arranged in a way that for a* k, $(k < n)$ *no* m, $(m = k + 1)$ *points selected arbitrary from them lie on a subspace with dimension less than* k*. Then these* m *points are said to be in general position.*

DEFINITEION 6.1.48. *Simplex : A convex closure of a set of points in a general position is said to be a simplex.*

Later we understand there are other restrictions on definition of a simplex.
We are not able to introduce the notion of isometry at this stage as we have not yet defined a length or as we understand from the word "isometry" a meter, or a norm for vectors. These need to be familiar with the idea of measures and metric spaces. Just assuming the reader is familiar with the "length" in ordinary Euclidean space we are going to have a first sight of some terminologies.

DEFINITEION 6.1.49. *Isometry (congruent tranaformation, or a congruence): Is a transformation on a space (on a set) that preserves length. There are two types of isometry: (1) direct isometry. (2) opposite isometry.*

Definiteion 6.1.50. *Identity Transformation : Is an isometry that leaves all points of space as invariant.*

Definiteion 6.1.51. *Rotation of Space : Is an isometry that leaves only one point as invariant. The invariant is called centre of rotation.*

Definiteion 6.1.52. *Translation of Space : Is an isometry that leaves no point as invariant.*

Definiteion 6.1.53. *Reflection of Space : Is an isometry that leaves a line as invariant. This invariant line is said to be the mirror for the reflection.*

Opposite isometry is product of odd number of reflections. Direct isometry is the product of even number of reflections. By product, we mean the product of transformation mappings.

Definiteion 6.1.54. *Inversion : Is an isometry f on $A \subseteq \Gamma$ such that $f(f(x)) = x$, $\forall x \in A$.*

Definiteion 6.1.55. *Half-turn (Reflection in a Point, Central Inversion) : Is a* 180° *rotation through the invariant point. Half-turn is an isometry.*

Definiteion 6.1.56. *Glide Reflection : This is combination of a reflection and a translation.*

Definiteion 6.1.57. *Symmetry : Symmetry transforms a shape into itself upon an isomery on the space.*

All the symmetries possible on a shape create a group under the binary operation of product of transformation, called group of symmetries of that shape. Any group of symmetries of a shape has its own generator.

In general we can have group of isometries of space; in simplest case group of isometries of plane.

6.2. Products of Groups

Remember that a group is always equipped with a binary operation. On different context we might call this binary operation as addition and talk about the sum of two elements of the group. Yet, in other occasions that might be called multiplication and we call the result of the binary operation product of its elements. Further, in some sets, we recognise two structure on the same elements co-existing side by side. Such is the situation, say, with, the set of integers. In such cases, we have to separate them by giving separate names for the underlayer structure binary operations: one called addition and the other multiplication. In such situations we might, additionally, discover new structures such as rings and fields.
Remember in ordinary arithmetics we have the addition group of integers $(\mathbb{Z}, +)$ and multiplicative group of integers $(\mathbb{Z}, \times)$ as two different things.

Also, we are going to have the notion of the **direct sum** of groups and the **free product** of groups. In their definition the **Cartesian product** of sets is also involved, that should be contrasted with other concepts.
Already we are familiar with a free subgroup of a group. Now we develop an artificial technique to make a free group out of a set. At the start assume we have a set $X = \{a, b, c, \cdots\}$. We are going to create a new set $\Gamma = \coprod_{i=1}^{2} X = \bigcup_{i=1}^{2} \{(x, i) \,|\, \forall x \in X\}$ which is disjoint union of X with X. Then carefully we turn Γ into a group.

DEFINITEION 6.2.1. *Free Inverse : Assume* (x, i) *and* (y, j) *in* Γ. *They are called inverse of each other if* $x = y$ *but* $i \neq j$. *We define* $(x) = (x, i)$ *and* $(x)^{-1} = (x, j)$. *We drop parenthesis to keep them simpler to use; hence,* $x = (x, i)$ *and* $x^{-1} = (x, j)$

DEFINITEION 6.2.2. *Free Word : A free word is a* ***finite*** *arbitrary selection of elements of* Γ *that allows* ***repeated*** *elements; such as* $\left(ace^{-1}cbaaadfecef^{-1}\right)$. *We can show any free word with an arbitrary symbol such as* $g = \left(ace^{-1}cbaaadfecef^{-1}\right)$.

Remark 6.2.1. In some context x and x^{-1} are called flip with respect to each other.

Therefore, a free word is an element taken from $\times_{i=1}^{n} \Gamma$ or Γ^n. You recognize that any word has a length of some n.

DEFINITEION 6.2.3. *Multiplication of Free Words : Assume* g_1 *and* g_2 *are two free words. Their multiplication is* $g_1 g_2$ *or* $g_2 g_1$. *That is to put the words next to each other (in juxtaposition).*

For example, $\left(ace^{-1}cbaaadfeef^{-1}\right)\left(dca^{-1}cbahcdfeeh^{-1}\right)$
$= \left(ace^{-1}cbaaadfeef^{-1}dca^{-1}cbahcdfeeh^{-1}\right)$
Conversely a word could be decomposed to smaller words; e.g., $\left(ace^{-1}cbaaadfeef^{-1}\right)$
$= \left(ace^{-1}c\right)\left(baa\right)\left(adfeef^{-1}\right)$
It is convenient to write the repeated words next to each other as powers; hence, $\left(ace^{-1}cbaaadfeef^{-1}\right) = \left(ace^{-1}cba^3dfe^2f^{-1}\right)$. Implicitly, we agreed that $a^m a^n = a^{m+n}$ for $a \in \Gamma$ and $m, n \in \mathbb{Z}$. Next we understand that, $a^0 = \not{e} = (\,)$. I use slashed $\not{e}$ in place of e or instead of using $\mathbf{1}$ for the unfortunate situations that Γ may include symbol e or symbol $\mathbf{1}$. As it is customary in universal algebra to use $(\,)$ for the empty list or the empty word, we also might be able to use it.

DEFINITEION 6.2.4. *Free Neutral Word : A neutral word is a word in the form* $\left(aa^{-1}\right)$ *or* $\left(a^{-1}a\right)$. *We show it by* $\not{e}$ *or* $(\,)$.

A neutral word has a zero length.

DEFINITEION 6.2.5. *Reduced Word : A word is reduced when all the neutral words in it are exhausted.*

Hence, $\left(ace^{-1}cbd^{-1}aa^{-1}dfeef^{-1}\right)$ is reduced to $\left(ace^{-1}cbd^{-1}dfeef^{-1}\right)$ and then to $\left(ace^{-1}cbfeef^{-1}\right)$.

DEFINITEION 6.2.6. *Free Group : The set G of all reduced words generated from Γ together with free multiplication is a group. This group is called the free group on X.*

Remark 6.2.2. Note that $x \in X$ can be identified with an $x \in G$ that is we can say $X^1 \subset G$. Most of the time study is limited to the case that $X = \{a\}$; that is X is a singleton.

DEFINITEION 6.2.7. *Generated Element : Assume $\{h_\alpha\}_{\alpha\in J}$ is a family of elements of G. Let $g = h_1 h_2 \cdots h_n$, where n is a finite number and h_j, $j = 1, \cdots, n$ could be repeated in the statement. Then g is said to be generated from the* ***generating*** *family $\{h_\alpha\}_{\alpha\in J}$.*

Set $h_1, h_2, \cdots, h_n$ means taking n arbitrary elements from the family $\{h_\alpha\}_{\alpha\in J}$. Sometimes, or frequently this selection is shown as $h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_n}$, rather, to emphasise on the index α.

DEFINITEION 6.2.8. *Group Generated by Subgroups : A group G is said to be generated by a family of its subgroup $H_{\alpha\in J}$ indexed by J whenever each element $g \in G$ can be expressed by a finite product of the elements of subgroups H_α. That is, $g = h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_n}$. The product $h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_n}$ is called a nomial or a term.*

Having been familiar with the notion of free words, now I define word in a more strict context. But it is easier to use them to create groups.

DEFINITEION 6.2.9. *Word : Assume G is a group generated by a family of its subgroup $H_{\alpha\in J}$ indexed by J. As said, any $g \in G$ can be expressed by multiplication of finite number of elements from subgroups H_α. An ordered set of these elements shown by $(h_{\alpha_1}, h_{\alpha_2}, \cdots h_{\alpha_n})$ is said to be a word of length n. Each h_{α_k}, in the word is called a factor.*

In some contexts a word is called a **list**. I do not use it, and spare **list** for usages in logic, automata, and computer theories. Note that one or more factors might belong to the same subgroup, say, H_{α_m}. Therefore, **each** $g \in G$ can be written as $h_1 h_2 \cdots h_k h_{k+1} \cdots h_n$, where h_k belongs to **some** H_{α_m}.
As we did **not** make the group G and its subgroups restricted to be Abelian (commutative), we cannot generally rearrange elements of each product.

DEFINITEION 6.2.10. *Reduced word: It could be that in $h_1 h_2 \cdots h_{k-1} h_k h_{k+1} \cdots h_n$ two or more factors next to each other such as $h_k h_{k+1}$ belong to the same H_α. This reduces the length of the word to $n-1$ or even smaller. This reduced size word is called a reduced word.*

Remark 6.2.3. All the subgroups generating G include an identity element e. In creating a word we do not participate this element. That always reduces the size of the word, being neutral. What if we have a word such as $(h_{\alpha_m}, h_{\alpha_m}^{-1})$. This reduces to a neutral (unity) element (e) and disappears. Hence, to take care of degenerated case, we define empty set as a word of length zero.
Consider reduced word of $g = g_1 g_2 \cdots g_m$ and the reduced word of $h = h_1 h_2 \cdots h_n$, then the word $gh = g_1 g_2 \cdots g_m h_1 h_2 \cdots h_n$ is a reduced word if g_m and h_1 both belong to the same subgroup.

Please note that the reduced word is a finite set, though there is no restriction on indexing set J to be finite and therefore the indexed family of subgroups could be infinite and even uncountable. To create a word we have to take factors from certain finite number of subgroups in the family. A certain subgroup might be used more than once, but not next to each other; in which case reduces the word length.

DEFINITEION 6.2.11. *Unity-joint Subgroups : Assume $H_{\alpha\in J}$ is a family of subgroups of G indexed by J. We say they are unity-joint if for $\alpha, \beta \in J$ and $\alpha \neq \beta$ we have $H_\alpha \cap H_\beta = e$, where e is the neutral element of G.*

In contrast to a (mutually) dis-joint family, the sets of this family are mutually joint in their common unity element. If it was not for the common element e they were mutually disjoint. Mutually hints to the fact that any pair you select from the family should have this property. In contrast, two subgroups H_α and H_β could have more than one element in common; in such case they are not unity-joint.

DEFINITEION 6.2.12. *Uniquely Represented Elements: Assume $g \in G$ and $H_{\alpha\in J}$ a unity-joint family of subgroups of G indexed by J. If we can factorize or show g, in terms of elements taken from H_α, only in one possible way, then we call it a uniquely represented element with respect to that family of subgroups. Further we assume that the related word is reduced. The neutral element $e \in G$ is defined as uniquely represented element.*

Note that we have **not** assumed G or any of its subgroups as an Abelian group.

DEFINITEION 6.2.13. *Free Subgroup of a Group : Assume A is a subset of the group G then the intersection F_A of all the subgroups of G that contain A is called the free subgroup of G over the set A.*

$$F_A = \bigcap_{\forall H_\alpha \subseteq G} H_\alpha \quad \text{such that} \quad A \subseteq H_\alpha \quad \text{and } H_\alpha \text{ a subgroup of } G.$$

DEFINITEION 6.2.14. *(Internal) Free Product Group : The set of all uniquely represented elements as defined in 6.2.12, taken from a unity-joint family $H_{\alpha\in J}$ of subgroups of G, make a subgroup of G. If the group G coincides with this subgroup , then it is said to be the free product of these subgroups. We show it by $G = \prod^*_{\alpha\in J} H_\alpha$.*

In the previous definition, we contrasted the group, by adding the modifier **free**, compared with the definition 6.2.8. I also added a modifier internal in the bracket to emphasise on the fact that the **sub**groups of the group are involved.

DEFINITEION 6.2.15. *External **Free** Product of Groups : Assume we have a group G and a family of groups $H_{\alpha\in J}$ indexed by J, **not** subgroups of G. Suppose the family of injections $i_\alpha : H_\alpha \longrightarrow G$ is a family of monomorphisms, such that G is the (internal) free product of $i_\alpha(H_\alpha)$ (check that they are unity-disjoint). Then, we say G is the external free product of the family H_α. We show it by $G = \prod^*_{\alpha\in J} H_\alpha$.*

Next we are going to define what is the direct product of groups. We start by informally explain it in the beautiful way that it first conceptually emerged as a group of mappings. Then the more modern definition will be presented.
Assume we have an indexed family of groups $\mathcal{H} = \{H_j, j \in J\}$. Define the set mapping $\alpha : \mathcal{H} \longrightarrow \bigcup \mathcal{H}$ such that $\alpha(H_j) = h_j$, where $h_j \in H_j$ for each group H_j. We are going to make a group G from the family α of all these mappings, by assuming the composition of mappings as the binary operation of group; that is, $\gamma = \alpha\beta$ means $\gamma(H_j) = (\alpha\beta)(H_j)$, and this means $\gamma(H_j) = \alpha(H_j)\beta(H_j)$. First define $\epsilon(H_j) = e_j$, where e_j is the unity (neutral) element of H_j. Then, inverse of a mentioned α is $\alpha^{-1} = \beta$ such that $\beta(H_j) = h_j^{-1}$; that is, $\alpha^{-1}\alpha = \alpha\alpha^{-1} = \alpha\beta(H_j) = h_j h_j^{-1} = e_j = \epsilon(H_j)$. This group is called the **full** direct product of the family of groups $\{H_j\}_{j\in J}$.
Now consider a subgroup G^* of G such that any $\alpha \in G^*$ assumes values $\alpha(H_j) = e_j$ for all, but for a finite number of j's. This is called the direct product of the family $\{H_j\}_{j\in J}$ of groups.
To further discover, let's single out for some $j_n \in J$ an $H_{j_n} \in \mathcal{H}$. Then take a subgroup shown as $N_{H_{j_n}}$ of all α's in G^* such that

$$\alpha_{j_n,h}(A) = \begin{cases} \epsilon(A) = e_A & \text{for} \quad A \in \mathcal{H}\backslash\{H_{j_n}\} \\ h & \text{for} \quad A = H_{j_n} \quad \text{where} \quad h \in H_{j_n} \end{cases}$$

This subgroup is normal in G^*.
To appreciate this fact, we need to show taking any $\beta \in G^*$ we get $\beta^{-1}\alpha\beta \in N_{H_{j_n}}$ for $\alpha \in N_{H_{j_n}}$. First, assume value of β for some $i \neq j$ is g_i and is also equal to h_j for H_{j_n}. Then $\beta^{-1}\alpha\beta$ is

$$(\beta^{-1}\alpha\beta)_{j_n,h}(A) = \begin{cases} \epsilon(A) = g_i^{-1} e_A g_i = e_A & \text{for} \quad A \in \mathcal{H}\backslash\{H_{j_n}\} \\ h_j^{-1} h h_j = h' & \text{for} \quad A = H_{j_n} \quad \text{where} \quad h' \in H_{j_n} \end{cases}$$

Then assume value of β for some $i \neq j$ is g_i and is e_j for H_{j_n}. Then $\beta^{-1}\alpha\beta$ is

$$(\beta^{-1}\alpha\beta)_{j_n,h}(A) = \begin{cases} \epsilon(A) = g_i^{-1} e_A g_i = e_A & \text{for} \quad A \in \mathcal{H}\backslash\{H_{j_n}\} \\ e_j^{-1} h e_j = h & \text{for} \quad A = H_{j_n} \quad \text{where} \quad h \in H_{j_n} \end{cases}$$

Therefore, $\beta^{-1}\alpha\beta$ is in $N_{H_{j_n}}$ and $N_{H_{j_n}}$ is a normal subgroup.
Show by $\mathcal{H}_N = \left\{N_{H_j}\right\}_{j\in J}$ the collection of all normal subgroups obtained in this way from each H_j, $j \in J$. When we have normal subgroups such as H_1 and H_2 then their set-multiplication, $H_1 H_2$, has a meaning. We have to overload usage of $\prod$ notation, regretfully, to be used for this set multiplication, too. At this point, question arises that, what is $\prod \mathcal{H}_N = \prod_{j\in J}\left\{N_{H_j}\right\}$? A little effort shows that this coincides with G^*. That is we could factor G^* into its normal subgroups. Remember G^* was a subgroup of the full direct product group G of the family of groups H_j. Now remove the normal subgroup $N_{H_{j_n}}$ corresponding to the singled out H_{j_n} from $\mathcal{H}_N$ and show the resulted collection by $\mathcal{H}_{\bar{H}_{j_n}}$ (please note that there is a bar over H to show that it is removed; that is, $\bar{H}_{j_n}$). Next, set-multiply the normal subgroups of this collection shown as $N^*_{j_n} = \prod \mathcal{H}_{\bar{H}_{j_n}}$ (had $N^*_{j_n}$ been a finite set-multiplication then $N^*_{j_n} = N_{j_1} N_{j_2} \cdots N_{j_{n-1}} N_{j_{n+1}} \cdots N_{j_m}$). Hence, $N^*_{j_n}$ is the set of all α's in G^* such that $\alpha(H_j) = \epsilon(H_j) = e_{H_j}$ for all $j \in J$.
At last, we succeed to define an isomorphism f_{j_n} from $H_{j_n} \in \mathcal{H}$ to the normal

subgroup $N_{H_{j_n}}$ of G^* as

$$f_{j_n}(h) = \alpha_{j_n,h}$$

Hence, G^* is the external free product of groups $H_{\alpha\in J}$.
Now, assume the family $\mathcal{N} = \left\{N^*_{j_n}\right\}_{j\in J}$. Then $\bigcap\mathcal{N} = \{\epsilon\}$
From this point on we do not use the modifier free anymore. You will see that no such context exists. We were free to repeatedly use the same G_α as far as the selected elements were not next to each other to reduce the length of the nomial set. Also we imposed a further restriction to unique representation of the words. After this point, we have Cartesian products of groups or our group are bound to be Abelian.

Definiteion 6.2.16. *External* ***Direct*** *Product of Groups : Assume we have a finite number of groups H_i, $i = 1, \cdots, n$, $n > 1$. Let $G = \times_{i=1}^n H_i$. Define a binary operation in G to multiply $g = (g_1, g_2, \cdots, g_n) \in G$ and $h = (h_1, h_2, \cdots, h_n) \in G$ by the rule $gh = (g_1, g_2, \cdots, g_n)(h_1, h_2, \cdots, h_n) = (g_1h_1, g_2h_2, \cdots, g_nh_n)$. With this definition G is a group if we define unit* ***y*** *(neutral) element as $e = (e_1, e_2, \cdots, e_n)$ and the unit (inverse) element of $g \in G$ as $g^{-1} = (g_1, g_2, \cdots, g_n)^{-1} = \left(g_1^{-1}, g_2^{-1}, \cdots, g_n^{-1}\right)$. We show it by $G = \prod_{i=1}^n H_i$.*

Definiteion was for the finite number of groups, created by the Cartesian product of an indexed family of groups.
It remains to show that the before mentioned definition is consistence with the previous discussion regarding full direct product of groups.
First remember definition 2.7.6 of Cartesian product for a family of sets. We showed that for a finite number of sets that definition coincides with the traditional definition of a Cartesian product.
Assume K is a set of mappings

$$\alpha : \{H_i \,|\, i = 1, \cdots, n,\ n > 1\} \longrightarrow \bigcup_{i=1}^{n} H_i,\ n > 1$$
$$H_i \longmapsto h \quad \text{where} \quad h \in H_i \quad \text{for each } i$$

Definiteion 6.2.17. *Internal Direct Product of Subgroups : Assume we have a finite number of subgroups H_i, $i = 1, \cdots, n$, $n > 1$ of group G. Then G is said to be the internal direct product of H_i, $i = 1, \cdots, n$, $n > 1$ of group G if there exist an isomorphism $\phi : \prod_{i=1}^n H_i \longrightarrow G$ such that $(h_1, h_2, \cdots, h_n)\,\phi = h_1h_2\cdots h_n$.*

It is possible to prove that with this definition each element $h \in G$ can uniquely be written as the product $h = h_1h_2\cdots h_n$, where $h_i \in H_h$. Hence internal direct product of subgroups is restriction of definition of free product of subgroups (6.2.14) from an indexed family of unknown number of subgroups to a finite number of subgroups.

Definiteion 6.2.18. *Internal (Abelian) Sum of Subgroups : Suppose G is an* ***Abelian*** *group and $\{H_\alpha\}_{\alpha\in J}$ is an indexed family of subgroups of G that generates G (see, 6.2.8). G is said to be the internal (Abelian) sum of the family of its subgroups $\{H_\alpha\}_{\alpha\in J}$. We show it by $G = \bigoplus_{\alpha\in J} H_\alpha$.*

Note that nomial representing any $g \in G$ in terms of $h_\alpha \in H_\alpha$ is a reduced nomial already, and has a unique participation of each H_α since G is **Abelian**.

DEFINITEION 6.2.19. *External (Abelian) Sum of Groups : Assume we have a group G and a family of groups $H_{\alpha\in J}$ indexed by J, not subgroups of G. Suppose the family of injections $i_\alpha : H_\alpha \longrightarrow G$ is a family of monomorphisms, such that G is the direct sum of $i_\alpha(H_\alpha)$ (check that they are unity-disjoint). Then, we say G is the external direct sum of the family H_α. This is the same as external **free** product of groups (6.2.15) but for the Abelian groups. For Abelian groups terminology uses "sum" instead of "product"*

DEFINITEION 6.2.20. *External Direct Sum of Groups : Assume we have a finite number of Abelian groups H_i, $i = 1, \cdots, n$, $n > 1$. Let $G = H_1 \times H_2 \times \cdots \times H_n$. Define a binary operation in G to multiply $g = (g_1, g_2, \cdots, g_n) \in G$ and $h = (h_1, h_2, \cdots, h_n) \in G$ by the rule $gh = (g_1, g_2, \cdots, g_n)(h_1, h_2, \cdots, h_n) = (g_1h_1, g_2h_2, \cdots, g_nh_n)$. This is the same as external **direct** product of groups (6.2.16) but for the Abelian groups. For Abelian groups terminology uses "sum" instead of "product"*

DEFINITEION 6.2.21. *Internal Direct Sum of Subgroups : Assume we have a finite number of subgroups H_i, $i = 1, \cdots, n$, $n > 1$ of Abelian group G. Then G is said to be the internal direct sum of H_i, $i = 1, \cdots, n$, $n > 1$ of group G if there exist an isomorphism $\phi : \prod_{i=1}^n H_i \longrightarrow G$ such that $(h_1, h_2, \cdots, h_n)\phi = h_1h_2 \cdots h_n$. This is the same as internal **free** product of subgroups (6.2.17) but for the Abelian groups. For Abelian groups terminology uses "sum" instead of "product"*

NOTATION 1. *External Free Product of Groups : This, **generally**, is shown by*

$$G = \prod_{\alpha\in J}^{*} H_\alpha$$

NOTATION 2. *External Abelian Sum of Groups : This, for **Abelian** groups, is shown by*

$$G = \bigoplus_{\alpha\in J}^{*} H_\alpha$$

*In such a context it is called external Abelian **sum** of the groups*

NOTATION 3. *External Direct Product of Groups : This, **generally**, is shown by*

$$G = \prod_{i=1}^{n} H_i$$

NOTATION 4. *External Direct Sum of Groups : This, for **Abelian** groups, is shown by*

$$G = \bigoplus_{i=1}^{n} H_i$$

*In such a context it is called external direct **sum** of the groups*

Remark 6.2.4. The following notation is reserved for **tensor** products

$$G = \bigotimes_{i=1}^{n} H_i$$

That could be applied to **tensor** product of Abelian groups too.

Remark 6.2.5. Summing up we have

(1) Group generated by certain indexed family of its subgroups (internal)
(2) (Internal) free product group generated by certain unity-joint indexed family of its own subgroups (union)
(3) External free product of (external family of) groups (monomorphism is involved) (union)
(4) External direct product of (external finite number of) groups (a new group is generated by Cartesian product)
(5) Internal direct product of subgroups (first we make the external direct product of subgroups then if isomorphism exists, a group already exist) (Cartesian product)
(6) Internal Abelian sum of subgroups (abelian similar to free product group generated by certain unity-joint indexed family of its subgroups) (union)
(7) External Abelian sum of (external family of) groups (union)
(8) External direct sum of groups (external direct product of groups.Cartesian but abelian a new group is generated) (Cartesian product)
(9) Internal direct sum of subgroups (first we make the external direct sum of subgroups then if isomorphism exists, a group already exist) (Cartesian product)
(10) Set product of subgroups is not generally a group except for Abelian groups
(11) Join of subgroups is a group

DEFINITEION 6.2.22. *Set Product of Subgroups : Set product of two subgroups H and K of G is defined as the set $HK = \{hk \mid \forall h \in H,$ and, $\forall k \in K\}$. This is not necessarily a subgroup.*

Take $a \in HK$ and $b \in HK$ then $a = h_1k_1$ and $b = h_2k_2$ and $ab = h_1k_1h_2k_2$. This, generally, cannot be expressed as hk for some $h \in H$ and some $k \in K$, except that G to be Abelian and $h_1k_1h_2k_2 = h_1h_2k_1k_2 = h_3k_3$. This can be restricted to the condition that if only elements of H and K, at least should **commute** with each other.

DEFINITEION 6.2.23. *Join (Least) of Subgroups : Assume H and K are two subgroups of G. Join or least of these two subgroup is shown by $H \vee K$ is defined as the intersection of all subgroups of G that contain the set $HK = \{hk \mid \forall h \in H,\ \forall k \in K\}$.*

Remark 6.2.6. A group G is the internal direct product of its subgroups H and K if and only if all these three conditions are satisfied.

(1) $G = H \vee K$
(2) $kh = hk \mid \forall h \in H$, and, $\forall k \in K$
(3) H and K are unity-disjoint; that is, $H \cap K = e$

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